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Time-dependent density profiles in a filling box

By M. GRAE WORSTER AND HERBERT E. HUPPERT
Department of Applied Mathematics and Theoretical Physics,
Silver Street, Cambridge CB3 9EW, England

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An approximate analytic expression for the time-dependent density profile formed by a turbulent buoyant plume in a confined region is presented. The analysis is based on the approximation that the density of the fluid behind the first front changes at a rate which is virtually independent of position. The approximate expression is shown to be in excellent agreement with a full numerical integration of the governing equations.

1. Introduction

Fluid that is initially homogeneous can become stably stratified due to the presence of isolated buoyancy sources. For example, a turbulent plume arising from a point source of buoyancy in a confined region can lead to stratification of the fluid surrounding the plume (Baines & Turner 1969). This is the situation to be considered here. It will be assumed that buoyancy forces are upward, which would be the case if the buoyancy was due to a heat source for example. Downward-flowing plumes may be treated by suitably inverting our solutions.

As a turbulent plume rises it entrains fluid from its environment. When the plume arrives at the top of the fluid container it spreads out laterally to form a layer of light fluid. The continuing plume now entrains light fluid from this layer and hence arrives at the top of the container even lighter. The plume thus spreads out above the existing light fluid layer, displacing the latter downwards. In this way a stratified region is produced, separated from the original unmodified fluid by a density step known as the first front.

Extending concepts first analysed by Morton, Taylor & Turner (1956), Baines & Turner (1969) developed expressions for the position of the first front as a function of time and for the large-time density gradient in the environment. This latter asymptotic state was calculated on the assumption that the density decreases linearly with time while all other fluid and flow properties remain constant.

A numerical scheme which follows the evolution of the stratified layer was developed by Germeles (1975). His work was motivated by a desire to understand the dynamic processes leading to an event, known as tank rollover, which can take place in storage tanks of liquefied natural gas. Stratification of the liquid gas can occur owing to input of new gas which may be of a different composition and temperature to that of the resident gas. The compositional and thermal gradients can lead to overturning (rollover) of the gas, causing large-scale boil-off of methane vapour. The resultant overpressure in the tank is potentially very dangerous.

There are other instances where knowledge of the density profile is required at times for which the asymptotic solution given by Baines & Turner is not applicable. For example, the property gradients in a filling box in which the fluid density is a function of two components (e.g. heat and salt) can lead to double-diffusive phenomena in the environment before the first front has advanced very far (McDougall 1983). The stratified region can break down into discrete well-mixed layers, and this layering can significantly affect the subsequent behaviour of the plume. In particular, the plume may no longer reach the top of the tank but can feed out into one of the underlying layers. While the mechanism producing the layers is not yet fully understood, it seems that the breakdown is governed by the opposing gradients of the two components. Since double-diffusive effects can be negligible in the plume while still significant in the environment (McDougall 1983), the results of the current paper are applicable to the initial development of the stratified region and can be used to evaluate the gradients of the two components before layering occurs.

The equations governing the behaviour of the plume are coupled nonlinear time-dependent partial differential equations. Complete analytic solutions are not known. The analysis presented here is based on the approximation that the time rate of change of density of the fluid in the stratified layer is virtually independent of position. The approximate solution, which is directly proportional to the asymptotic solution with a constant of proportionality that depends on time, has the following properties. It is correct at $t = 0$, it tends to the asymptotic solution found by Baines & Turner in the limit $t \to \infty$, and global conservation of buoyancy is satisfied. Finally, the approximate solution is shown to be in excellent agreement with a full numerical integration of the governing equations.

The strength of the solution lies in the accuracy with which it fits the numerical results and in its simple relation to the asymptotic solution given by Baines & Turner. It may thus be applied with confidence and ease in many practical situations.

2. The governing equations

The derivation of the basic equations has been discussed in detail elsewhere (e.g. Turner 1979). Consider a tank of fluid with uniform cross-sectional area A, minimum horizontal dimension L and height H. The aspect ratio L/H is assumed to be sufficiently large that the vertical velocity in the environment is horizontally uniform (Baines & Turner 1969; Huppert & Sparks 1983). A point source of buoyancy, of constant flux F_0, is placed at the bottom of the tank, well away from the vertical sidewalls (figure 1). The resulting axisymmetric plume is assumed to have Gaussian profiles of vertical velocity and of buoyancy given by

$$w(z, r) = w(z) \exp(-r^2/b^2), \tag{1}$$

$$g[\rho_0(z) - \rho(z, r)]/\rho_1 = \Delta(z) \exp(-r^2/b^2), \tag{2}$$

where the z-axis is positive upwards, r is the radial coordinate from the plume axis, ρ and ρ_0 are the densities inside and outside the plume, ρ_1 is some reference density, and g is the acceleration due to gravity. The dependent variables which will appear in the subsequent analysis are: w, the vertical velocity at the plume axes; Δ, the maximum buoyancy in the plume at each height; and b, the width of the plume defined as the radial distance at which the values of w and Δ have fallen to $1/e$ of their maximum values. Rouse, Yih & Humphreys (1952) showed that Gaussian profiles, such as (1) and (2), give good fits to experimental measurements of turbulent plumes, though the width of the profiles for the velocity and density fields are usually

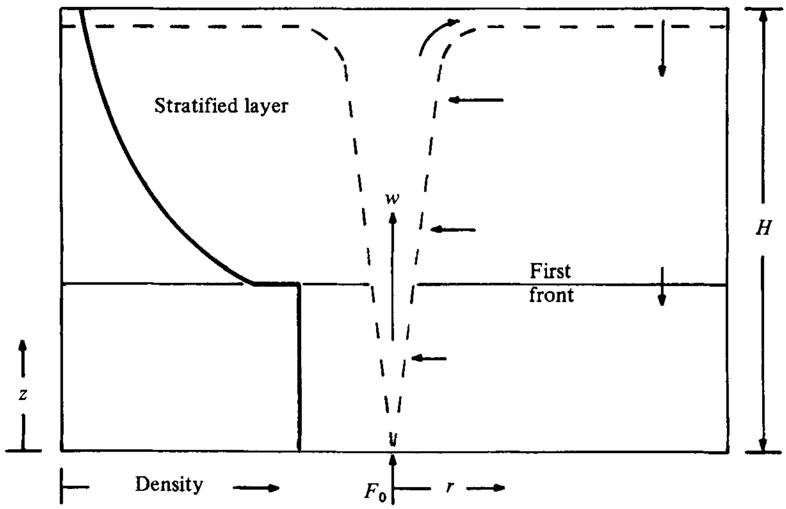

FIGURE 1. A sketch of the filling box some time after introduction of the point heat source. The right hand side indicates the fluid motion. The left-hand side shows the density profile above and below the first front.

different. Two points should be noted, however. First, it is straightforward to incorporate the difference in spread of the two profiles into the analysis (Morton 1959). Secondly, the exact analytic form chosen for the profiles, (1) and (2), does not affect the conclusions made here since all the results are expressed in terms of mean values of the dependent variables, averaged with respect to the volume flux in the plume. For example, an often-used approximation is to assume 'top-hat' profiles (Turner 1979, chap. 6); i.e. the vertical velocity and density deficit are assumed to have constant values for $0 \leqslant r \leqslant b$ and to be zero for $r > b$.

The Boussinesq approximation and the entrainment assumption are employed in deriving equations representing conservation of volume, momentum and buoyancy. These are respectively

$$\frac{\mathrm{d}}{\mathrm{d}z}(b^2 w) = 2\alpha b w, \tag{3}$$

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{b^2 w^2}{2}\right) = b^2 \Delta, \tag{4}$$

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{b^2 w \Delta}{2}\right) = b^2 w \frac{\partial \Delta_0}{\partial z}. \tag{5}$$

The Boussinesq approximation is valid when density variations are sufficiently small that they have negligible effect on the inertia of the fluid. It is usually assumed that the entrainment constant α is not a function of any external variables, and we shall invoke this assumption in this paper. Molecular diffusion in the environment is also usually neglected. The evolution of the buoyancy field in the environment Δ_0 is thus given by

$$\frac{\partial \Delta_0}{\partial t} = \frac{\pi b^2 w}{A} \frac{\partial \Delta_0}{\partial z}, \tag{6}$$

where $\Delta_0 = g(\rho_0 - \rho_1)/\rho_1$. For convenience we introduce the non-dimensionalized variables

$$\zeta = H^{-1}z, \quad \tau = 4\pi^{\frac{2}{3}}\alpha^{\frac{4}{3}}H^{\frac{2}{3}}A^{-1}F_0^{\frac{1}{3}}t,$$

$$\delta = 4\pi^{\frac{2}{3}}\alpha^{\frac{4}{3}}H^{\frac{5}{3}}F_0^{-\frac{2}{3}}\Delta_0, \quad f = \tfrac{1}{2}\pi F_0^{-1}b^2 w\Delta,$$

$$q = \tfrac{1}{4}\pi^{\frac{1}{3}}\alpha^{-\frac{4}{3}}H^{-\frac{5}{3}}F_0^{-\frac{1}{3}}b^2 w, \quad m = \tfrac{1}{2}\pi^{\frac{1}{3}}\alpha^{-\frac{1}{3}}H^{-\frac{2}{3}}F_0^{-\frac{1}{3}}bw.$$

The new dependent variables f, q and m^2 represent the fluxes in the plume of buoyancy, volume and momentum, and δ represents the buoyancy of the environment. The new independent variables ζ and τ are the non-dimensional height and time respectively. Note that the non-dimensionalized buoyancy flux f is equal to unity at the bottom of the container, $\zeta = 0$. In terms of these new variables, (3)–(5) may be written

$$\frac{\mathrm{d}q}{\mathrm{d}\zeta} = m, \quad \frac{\mathrm{d}m^4}{\mathrm{d}\zeta} = 2qf, \quad \frac{\mathrm{d}f}{\mathrm{d}\zeta} = q\frac{\partial \delta}{\partial \zeta}, \qquad (7a, b, c)$$

while (6) may be written

$$\frac{\partial \delta}{\partial \tau} = q\frac{\partial \delta}{\partial \zeta}. \qquad (8)$$

3. Boundary conditions for the stratified layer

In a uniform unconfined environment the right-hand side of (7c) is zero, and a similarity solution exists in which

$$f = 1, \qquad (9)$$

$$q = \tfrac{3}{10}(\tfrac{18}{5})^{\frac{1}{3}}\zeta^{\frac{5}{3}}, \qquad (10)$$

$$m = \tfrac{1}{2}(\tfrac{18}{5})^{\frac{1}{3}}\zeta^{\frac{2}{3}}. \qquad (11)$$

It is assumed that these expressions remain valid in a confined environment for $\zeta < \zeta_0$, where ζ_0 is the position of the first front. The density step across the first front is given by the mean density of the plume, averaged with respect to the volume flux, when it first reaches the top of the container. This is $\frac{1}{2}\Delta$ evaluated at $z = H$, and may be expressed in non-dimensional form as

$$-\delta_0 = \frac{f}{q}\bigg|_{\zeta=1} = \tfrac{10}{3}(\tfrac{5}{18})^{\frac{1}{3}} \approx 2.175. \qquad (12)$$

The position of the first front at any later time may be found by considering the global volume conservation in the container. This requires

$$\frac{\mathrm{d}\zeta_0}{\mathrm{d}\tau} = -q|_{\zeta_0} = -\tfrac{3}{10}(\tfrac{18}{5})^{\frac{1}{3}}\zeta_0^{\frac{5}{3}},$$

which can be integrated to yield

$$\zeta_0 = [1 + \tfrac{1}{5}(\tfrac{18}{5})^{\frac{1}{3}}\tau]^{-\frac{3}{2}}. \qquad (13)$$

The solution to (7) and (8), valid as $\tau \rightarrow \infty$, assumes that $\partial\delta/\partial\tau = 1$ independent of ζ. The dependent variables may then be calculated to be

$$\left.\begin{aligned} q = q_\infty &\equiv \tfrac{3}{10}(\tfrac{18}{5})^{\frac{1}{3}}\zeta^{\frac{5}{3}}\{1-\tfrac{5}{39}\zeta-\tfrac{265}{12168}\zeta^2+\ldots\}, \\ m = m_\infty &\equiv \tfrac{1}{2}(\tfrac{18}{5})^{\frac{1}{3}}\zeta^{\frac{2}{3}}\{1-\tfrac{8}{39}\zeta-\tfrac{583}{12168}\zeta^2+\ldots\}, \\ f = f_\infty &\equiv 1-\zeta, \\ \tau+\delta = \delta_\infty &\equiv 5(\tfrac{5}{18})^{\frac{1}{3}}\zeta^{-\frac{2}{3}}\{1-\tfrac{10}{39}\zeta-\tfrac{155}{8112}\zeta^2+\ldots\}. \end{aligned}\right\} \qquad (14)$$

These expressions are identical with those given by Baines & Turner (1969). It is readily shown from (7*c*) and (12) that the value of f just above the position of the first front is

$$f = f_0(\tau) \equiv 1-\zeta_0^{\frac{5}{3}}. \qquad (15)$$

It is assumed that when the fluid in the plume reaches the top of the container it spreads instantaneously to form an upper layer of light fluid. The density of the fluid in this layer must therefore be equal to the mean density of the plume at the instant just before the plume spreads out. Hence, at $\zeta = 1$,

$$f(1,\tau) = 0. \qquad (16)$$

It might appear that the five boundary conditions (10)–(12), (15) and (16) overspecify the fourth-order system (7), (8). However, (7*c*) and (8) can be combined to give

$$\frac{\mathrm{d}f}{\mathrm{d}\zeta} = \frac{\partial\delta}{\partial\tau}. \qquad (17)$$

This can be integrated from $\zeta = 0$ to $\zeta = 1$ to yield

$$[f]_{\zeta=0}^{\zeta=1} = \frac{\mathrm{d}}{\mathrm{d}\tau}\int\delta\,\mathrm{d}\zeta = 1 \qquad (18)$$

since $\int\delta\,\mathrm{d}\zeta$ represents the total buoyancy in the container and increases linearly with time. Thus (16) appears as an integral constraint on the governing equations and must hold in order to satisfy the global conservation of buoyancy.

4. The approximate solution

Knowing the value of f at $\zeta = \zeta_0$ and at $\zeta = 1$ from (15) and (16), we invoke the approximation that f is linear between these points. This is equivalent to assuming that the rate of change with time of the density of the fluid above the first front is independent of position at leading order in ζ_0 (cf. (17)). Thus

$$f(\zeta,\tau) = \hat{f}(\tau)\,(1-\zeta), \qquad (19)$$

where

$$\hat{f}(\tau) = \frac{1-\zeta_0^{\frac{5}{3}}}{1-\zeta_0}. \qquad (20)$$

With this approximation, it is readily seen that the solution of (7) is

$$q = \hat{f}^{\frac{1}{3}}(\tau)\,q_\infty(\zeta), \quad m = \hat{f}^{\frac{1}{3}}(\tau)\,m_\infty(\zeta), \quad \delta = \hat{f}^{\frac{2}{3}}(\tau)\,\delta_\infty(\zeta)-c(\tau). \qquad (21\,a,b,c)$$

Choosing the constant of integration $c(\tau)$ to satisfy the integral constraint (18), we obtain

$$c(\tau) = 5\left(\frac{5}{18}\right)^{\frac{1}{3}}\left\{\frac{\zeta_0^{-\frac{2}{3}}-1}{1-\zeta_0}+3\hat{f}^{\frac{2}{3}}\left[\frac{1-\zeta_0^{\frac{1}{3}}}{1-\zeta_0}-\frac{5}{78}\frac{1-\zeta_0^{\frac{4}{3}}}{1-\zeta_0}-\frac{155}{56784}\frac{1-\zeta_0^{\frac{7}{3}}}{1-\zeta_0}+\ldots\right]\right\}. \qquad (22)$$

While the expression for $c(\tau)$ appears rather cumbersome, it should be noted that in most circumstances it is only the density gradient and not the actual density which is required. In this case $c(\tau)$ is an unnecessary parameter. It is readily shown that, as $\tau \to 0$ ($\zeta_0 \to 1$), $\delta \to \delta_0$, the value of the density step across the first front. In addition, since $\hat{f} \to 1$ as $\tau \to \infty$, the expressions (21) tend to the asymptotic solutions (14) in this limit.

A formal asymptotic expansion (in powers of ζ_0) for q, m and δ could be obtained by using (8) or (17) as a pivot for iteration. Equation (19) could be treated as the first iterate, then, at each order, q, m and δ could be determined from (7) and f updated using either (8) or (17). In such a way solutions to the governing equations could be determined to arbitrary accuracy. However, it is felt that to develop the expansions further than (21) would not be justifiable in view of the intrinsic approximations already made in developing (3)–(6), and would detract from the simplicity of these expressions for use in practical situations.

The degree to which (21) does not satisfy (17) determines the formal error in the approximate solution. Note first that from (13)

$$\frac{d\zeta_0}{d\tau} = -\tfrac{3}{10}\left(\tfrac{18}{5}\right)^{\frac{1}{3}}\zeta_0^{\frac{5}{3}} = O(\zeta_0^{\frac{5}{3}}).$$

From (19) and (20)

$$\frac{df}{d\zeta} = -\frac{1-\zeta_0^{\frac{5}{3}}}{1-\zeta_0} = -[1+\zeta_0+O(\zeta_0^{\frac{5}{3}})],$$

while from (21) and (22)

$$\begin{aligned}\frac{\partial\delta}{\partial\tau} &= \tfrac{2}{3}f^{-\frac{1}{3}}\frac{df}{d\zeta_0}\frac{d\zeta_0}{d\tau}f_\infty(\zeta)-\frac{dc}{d\tau} \\ &= O(\zeta_0^{\frac{5}{3}})f_\infty(\zeta)-(1+\zeta_0+O(\zeta_0^{\frac{5}{3}})) \\ &= \frac{df}{d\zeta}+O(\zeta_0^{\frac{5}{3}}).\end{aligned}$$

Thus, as $\zeta_0 \to 0$, the error in (17) is formally $O(\zeta_0^{\frac{5}{3}})$.

5. A numerical solution

A numerical scheme, using an approach similar to that used by Germeles (1975), was applied to (7) and (8). In this scheme the density of the environment is represented by a stepped profile. At each timestep (7*c*) is integrated analytically to give the corresponding stepped profile for f while (7*a*, *b*) are integrated numerically by a Runge–Kutta integration scheme. Finally the method of characteristics applied to (8) determines how the density levels (the positions of the steps) move with time. The results of the numerical integration are shown graphically in figures 2–5.

The density profile predicted by (21) is superimposed on the numerical result in figure 2. As can be seen, there is very close agreement between the numerical and analytic solutions. In fact the discrepancy is less than 1 % over most of the range. Figure 3 shows that the linear approximation made for f, as expressed by (19), is reasonable. The agreement between the theoretical and numerical calculations of the volume and momentum fluxes is not so good, with maximum discrepancies of about

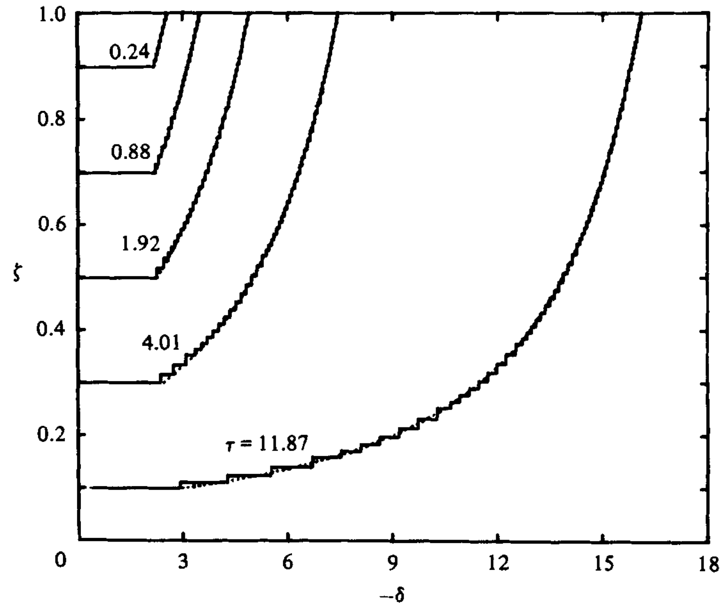

FIGURE 2. The density profile δ as a function of height ζ at various times τ. The variables δ, ζ and τ are non-dimensional. The solid lines are given by the numerical solution; the dotted lines by the analytic solution. The steps in the numerical solution are an artefact of the numerical scheme and are not meant to imply a stepped density profile in an actual filling box.

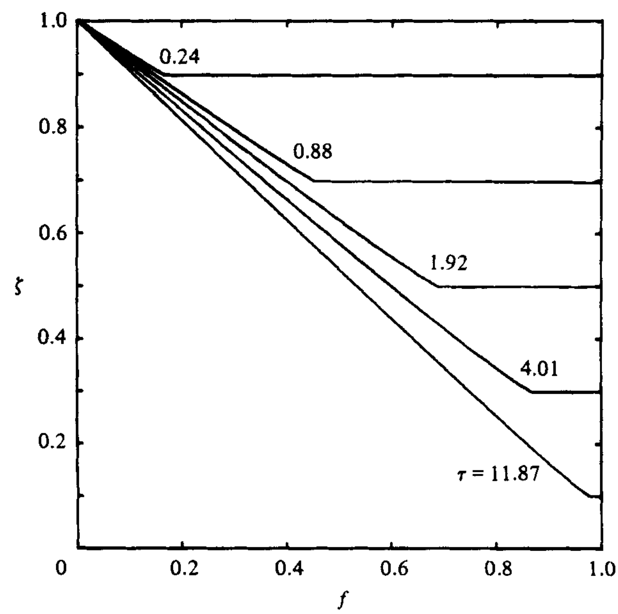

FIGURE 3. Numerical solution for the buoyancy flux f in the plume, as a function of height ζ at various times τ. The variables f, ζ and τ are non-dimensional.

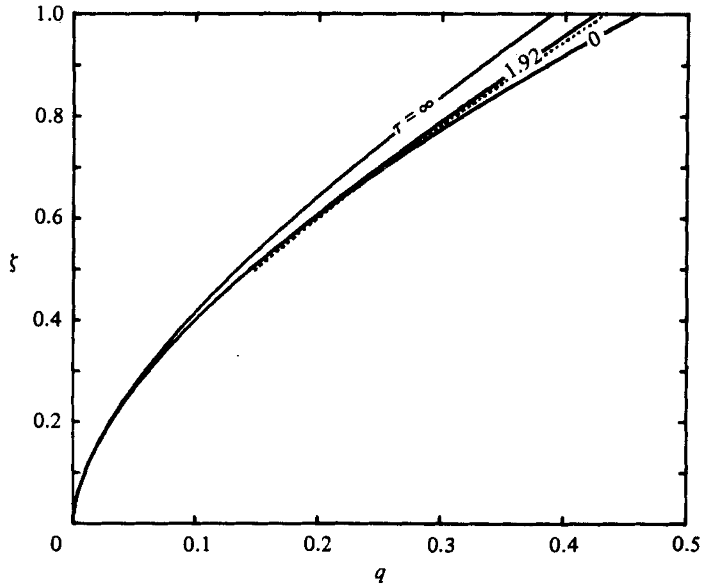

FIGURE 4. Numerical solution (solid line) and analytic solution (dotted line) for the volume flux q as a function of height ζ at time $\tau = 1.92$. The variables q, ζ and τ are non-dimensional. The curves for $\tau = 0$ and $\tau = \infty$ are the similarity solution (10) and the asymptotic state (14) respectively.

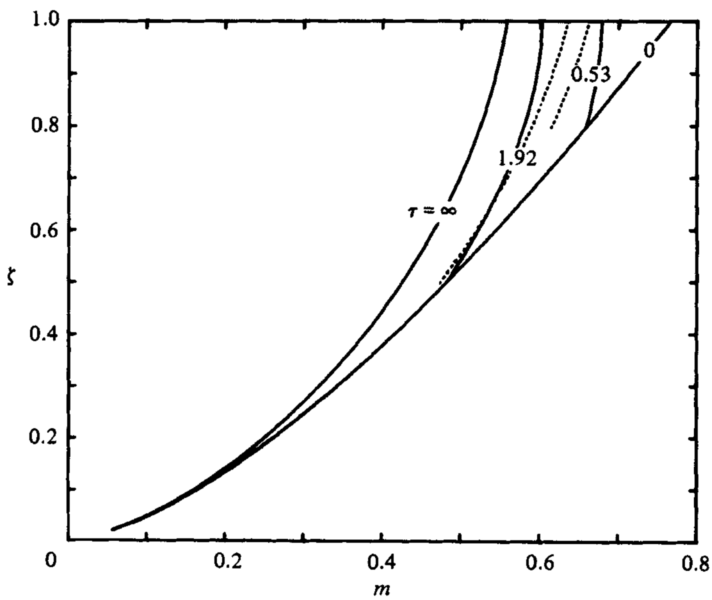

FIGURE 5. Numerical solutions (solid lines) and analytic solutions (dotted lines) for the square root of the momentum flux m as a function of height ζ at times $\tau = 0.53$ and $\tau = 1.92$. The variables m, ζ and τ are non-dimensional. The curves for $\tau = 0$ and $\tau = \infty$ are the similarity solution (11) and the asymptotic state (14) respectively. The fit between the numerical and analytic curves improves as $\tau \rightarrow \infty$. The only error in that limit is due to truncation of the power-series expansion for the asymptotic state (14).

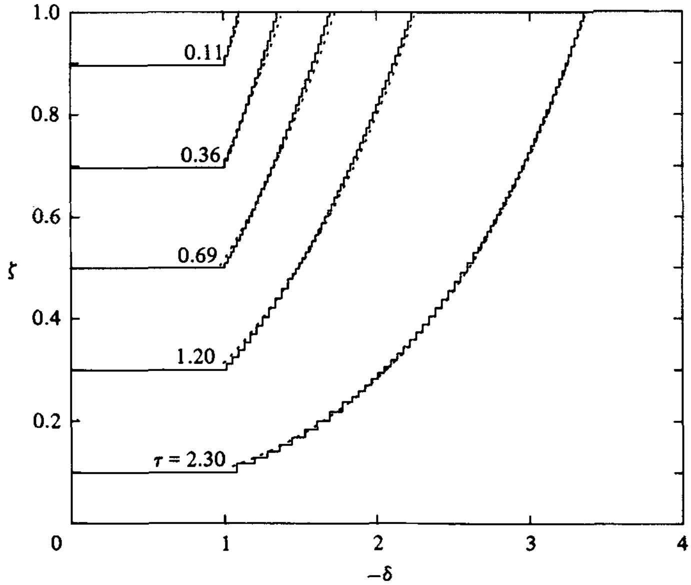

FIGURE 6. The density profile δ as a function of height ζ at various times τ for a two-dimensional filling box. The variables δ, ζ and τ are non-dimensional. The solid lines are given by a numerical solution; the dotted lines by the analytic solution.

3 % and 8 % respectively. Some of this discrepancy is attributable to the inaccuracy with which the first three terms of the power series expansions of q_∞ and m_∞ describe the actual asymptotic solution. The fit is less good for small τ (large ζ_0) since the formal error in the analytic expressions is $O(\zeta_0^{\frac{5}{3}})$, which increases as ζ_0 increases.

6. The line plume

The evolution of the stratification in a two-dimensional filling box with buoyancy supplied at a line source may be analysed by employing the same approximation as above. In this case, however, the time-dependent factor $\hat{f}(\tau)$ is unity. Thus the asymptotic solution found by Baines & Turner (1969) may be used directly to approximate the density field above the first front. The solution is presented here for completeness, and is given by

$$\delta = \delta_\infty(\zeta) - c(\tau), \tag{23}$$

where

$$\delta_\infty(\tau) = -\ln\zeta - \tfrac{1}{8}\zeta - \tfrac{11}{640}\zeta^2 + \dots, \tag{24}$$

$$c(\tau) = -\ln\zeta_0 + 1 - \tfrac{1}{16}(1+\zeta_0) - \tfrac{11}{1920}(1+\zeta_0+\zeta_0^2) + \dots, \tag{25}$$

$$\zeta_0 = \mathrm{e}^{-\tau}. \tag{26}$$

The non-dimensionalization employed here is the same as that used by Baines & Turner (1969). The error in this solution is formally $O(\zeta_0)$, and figure 6 shows that, while the agreement with the numerical solution is not quite so good as that found in the case of the axisymmetric plume, it is more than adequate for most purposes.

7. Conclusion

An analytic expression has been found which closely approximates the density profile in a filling box as it evolves with time. This is an improvement over existing methods of determining the time evolution of filling boxes which relied on numerical simulations and hence required separate evaluation for each time period of interest.

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REFERENCES

Baines, W. D. & Turner, J. S. 1969 Turbulent buoyant convection from a source in a confined region. *J. Fluid Mech.* **37**, 51–80.

Germeles, A. E. 1975 Forced plumes and mixing of liquids in tanks. *J. Fluid Mech.* **71**, 601–623.

Huppert, H. E. & Sparks, R. S. J. 1983 The effect of aspect ratio on the behaviour of a filling box. (In preparation.)

McDougall, T. J. 1983 Double-diffusive plumes in confined environments. *J. Fluid Mech.* (to be published).

Morton, B. R. 1959 Forced plumes. *J. Fluid Mech.* **5**, 151–163.

Morton, B. R., Taylor, G. I. & Turner, J. S. 1956 Turbulent gravitational convection from maintained and instantaneous sources. *Proc. R. Soc. Lond.* **A234**, 1–23.

Rouse, H., Yih, C. S. & Humphreys, H. W. 1952 Gravitational convection from a boundary source. *Tellus* **4**, 201–210.

Turner, J. S. 1979 *Buoyancy Effects in Fluids*. Cambridge University Press.