The Axisymmetric Laminar Plume: Asymptotic Solution for Large Prandtl Number

By M. Grae Worster

The method of matched asymptotic expansions is applied to the axisymmetric boundary-layer equations in order to determine approximate solutions for free convection from a point source of buoyancy in the limit of large Prandtl number ($\sigma \gg 1$). In common with other types of free-convection boundary layers at large Prandtl number, there is an inner region in which the temperature decays to its far-field value, and a much wider, outer region in which the vorticity decays to zero. Unlike the other cases, the velocity is not of the same order of magnitude in the two regions, but is larger in the inner region by a factor of order $\ln(1/\epsilon^2)$ in the inner region, where ϵ is a root of $\epsilon^4 \ln(1/\epsilon^2) = 1/\sigma$.

1. Introduction

A point source of buoyancy in an unbounded fluid can give rise to natural convection in the form of a narrow vertical plume [9]. Most plumes in common experience are turbulent (e.g. smoke from a chimney), but there are circumstances, typically when the fluid viscosity is large, in which a plume will remain laminar for a considerable height.

A growing interest in the dynamics of various geological phenomena, especially the fluid mechanics of molten rock, has inspired investigations into the flow of extremely viscous fluids. In particular, there is a need to understand natural convective flows when the Prandtl number $\sigma = \nu/\kappa$ (where ν is the kinematic viscosity and κ is the thermal diffusivity) is very large, since Prandtl numbers of geological fluids can vary from about 10^2 for a hot basalt to about 10^{23} for the earth's mantle. Convective flows driven by compositional variations are also important in geological contexts, and these can be described by the same

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analyses as describe thermal convection once the Prandtl number is replaced by the Schmidt number ν/D, where D is the molecular diffusivity of the buoyant component. Schmidt numbers are typically larger even than the Prandtl number, owing to the very slow diffusion of most chemical species compared with the diffusion of heat.

When the Prandtl number is large, the length scale for thermal variations is much smaller than that for variations of vorticity. Therefore, most analyses of free convection at large Prandtl number are of boundary-layer type with thermal buoyancy balancing viscous dissipation in the inner region. If the Prandtl number is sufficiently large (as it is in the mantle), then the outer flow is just Stokes flow driven by viscous shear stresses generated by the inner, thermal layer [7, 6]. In other words, inertia can be safely neglected throughout the whole flow field. However, in other situations, such as the input of light fluid from the base of a magma chamber [4], inertia can play a significant role, an outer boundary layer exists in which viscous dissipation is balanced by inertia, and the very outer fluid is stationary.

The aim of this paper is to investigate axisymmetric, laminar, convective plumes, at large Prandtl number, rising through an infinite, stationary fluid region. Thereby, we shall determine the width of the vorticity boundary layer and thus be able to determine a criterion for the safe neglect of inertia. In situations where this width is much less than the width of the natural fluid region, and inertia may not be neglected, we shall obtain asymptotic expressions for important properties of the flow such as the total convective volume flux and the center line temperature and velocity of the plume.

There are many investigations of two-dimensional and axisymmetric laminar plumes in the literature, beginning with Zel'dovich [13]; see Yih [12] for a review of past work. However, the few analyses of their behavior at large Prandtl number have dealt only with the two-dimensional plume [8, 5]. Although many of the same principles apply equally to the case of an axisymmetric plume, and the analytical technique (matched asymptotic expansions) used in this paper is not new, the details of the calculation were not found to be straightforward. In particular, the expansion parameters could not be guessed using the physical arguments that apply to a two-dimensional plume, and they are determined only by matching constraints at second order in the expansions.

In Section 2 the similarity transformation of the boundary-layer equations is introduced and numerical solutions are presented for various finite values of the Prandtl number up to $\sigma = 10$. Asymptotic expansions for the inner, thermal boundary layer and the outer, momentum boundary layer are found in Section 3, and are matched by the introduction of an intermediate variable. Several of the matching constants are found analytically; the remainder are found numerically by integrating the scaled equations. The concluding section draws attention to the principal results and discusses the limits of applicability of the solutions.

2. The equations and similarity transformation

Consider steady, laminar, free convection from a point source of heat in an unbounded, homogeneous, Newtonian fluid with constant properties. We seek a flow that is significant only in a narrow, vertical region above the heat source.

Therefore, we adopt the usual boundary-layer approximation [2] that radial variations are much more rapid than those in the longitudinal direction. This approximation needs *post hoc* justification, which it will receive in the conclusion. The radial component of the Navier-Stokes equation shows that the nonhydrostatic pressure is constant everywhere, so the longitudinal component can be expressed as

$$u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z} = \nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + \frac{\partial^2 w}{\partial z^2}\right] + \alpha g(T - T_0), \tag{2.1}$$

in terms of cylindrical polar coordinates r (radial) and z (vertical and upwards), where α is the coefficient of thermal expansion and g is the acceleration of gravity. The velocity components corresponding to (r, z) are (u, w) and are assumed to be axisymmetric. The last term of (2.1) is the buoyancy force due to the difference between the local temperature T and the temperature of the ambient fluid, T_0. Note that the temperature field can be replaced by any diffusing agent that causes variations in the fluid density. It is the buoyancy term which provides the coupling between the Navier-Stokes equation and the diffusion-advection equation

$$u\frac{\partial T}{\partial r} + w\frac{\partial T}{\partial z} = \kappa\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{\partial^2 T}{\partial z^2}\right]. \tag{2.2}$$

In these equations the Boussinesq approximation has been employed: variations in density are ignored except insofar as they modify the buoyancy, and all other physical properties of the fluid are taken as constant. The boundary conditions are that the vertical shear and the radial temperature gradient vanish on the axis of symmetry $r = 0$, and that the vertical velocity w and the temperature perturbation $T - T_0$ vanish as $r \to \infty$. In addition, there is an integral constraint, expressing conservation of the vertical heat flux, given by

$$\rho C_p \int_0^\infty w(T - T_0)\, 2\pi r\, dr = Q, \tag{2.3}$$

where ρ is the fluid density, C_p is its specific heat, and Q is the constant heat flux of the point source. Mass conservation is assured by the introduction of a Stokes steam function $\psi(r, z)$ such that

$$u = -\frac{1}{r}\frac{\partial \psi}{\partial z}, \qquad w = \frac{1}{r}\frac{\partial \psi}{\partial r}. \tag{2.4}$$

Within the boundary-layer approximation, the vertical diffusion terms in (2.1) and (2.2) are ignored, and, since there is no externally imposed lengthscale in the problem, the remaining equations admit a similarity solution of the form

$$\psi = \nu z F(\zeta), \tag{2.5}$$

$$T - T_0 = \frac{Q}{kz} h(\zeta), \tag{2.6}$$

$$\zeta = G^{1/4} r/z, \tag{2.7}$$

$$G = \alpha g \frac{Q}{k} \frac{z^2}{\nu^2}, \tag{2.8}$$

where $k = \rho C_p \kappa$ is the thermal conductivity of the fluid, and G is a local, modified Grashof number. This formulation is similar to that given by Yih [11] and by Brand and Lahey [3]. Note that G is not a constant parameter but is a function of z. We shall see that G needs to be large in order for the boundary-layer analysis to be valid. We shall then have an adequate description of the plume provided $G_L = \alpha g(Q/k)L^2/\nu^2$ is large, where G_L is the (constant) modified Grashof number based on the vertical dimension L of the system. The dimensionless functions F and h satisfy ordinary differential equations

$$[\zeta(F'/\zeta)']' = -\zeta h - F(F'/\zeta)', \tag{2.9}$$

$$h' = -\frac{\sigma}{\zeta} Fh, \tag{2.10}$$

$$2\pi \int_0^\infty F'h \, d\zeta = \frac{1}{\sigma}, \tag{2.11}$$

subject to the boundary conditions

$$F = 0, \qquad (F'/\zeta)' = 0, \qquad (\zeta = 0), \tag{2.12}$$

$$h \to 0, \qquad F'/\zeta \to 0 \qquad (\zeta \to \infty). \tag{2.13}$$

Yih [11] and Brand and Lahey [3] independently found closed-form solutions to (2.9)–(2.13) for the special cases $\sigma = 1$ and $\sigma = 2$, and Brand and Lahey integrated the equations numerically for various values of σ up to $\sigma = 10$. The numerical results of Brand and Lahey are inconsistent with the asymptotic results found later in this paper, and so I recomputed the solutions for some finite values of σ. I believe my numerical results, given in Table 1, to be accurate to the number of significant figures shown. The solutions decay algebraically as $\zeta \to \infty$, so a nonlinear coordinate transformation

$$\omega = \ln \zeta \tag{2.14}$$

was applied before integrating the equations using a fourth-order Runge-Kutta

Table 1
Nondimensional Volume Flux F_∞, Centerline velocity W, and Centerline Temperature H for an Axisymmetric Laminar Plume at Various Values of the Prandtl Number σ [a]

σ	F_∞	W	H
1	6.000	0.3989	0.1061
2	4.000	0.3154	0.0995
5	3.398	0.2279	0.0924
10	3.309	0.1768	0.0896
∞	3.267	$0.3989\epsilon^2 \ln \epsilon^{-2}$	0.0796
		$+0.0018\epsilon^2 + \cdots$	$+0.0122(\ln \epsilon^{-2})^{-1} + \cdots$

[a] See Section 3 for the derivation of the infinite-Prandtl-number results.

scheme with variable step lengths. A shooting method was employed which started from the asymptotic expansions

$$F \sim \tfrac{1}{2}We^{2\omega} - \tfrac{1}{16}He^{4\omega},$$

$$h \sim H - \tfrac{1}{4}\sigma HWe^{2\omega} + \left(\tfrac{1}{64}\sigma H^2 + \tfrac{1}{32}\sigma^2 HW^2\right)e^{4\omega},$$

applied at some $\omega = \omega_0 < 0$ and integrated to some $\omega = \omega_1 > 0$. The constants W and H, which respectively represent the vertical velocity and the temperature on the plume axis, were adjusted until the boundary conditions (2.13) were satisfied at $\omega = \omega_1$. The range for ω in the differential equations is $-\infty < \omega < \infty$, so the numerical boundary positions ω_0 and ω_1 were systematically decreased and increased respectively until their varying made no appreciable change to the computed results. Graphs of the vertical velocity and the temperature are shown in Figure 1.

3. Asymptotic expansions for large Prandtl number

The results of the previous section show that as the Prandtl number becomes large a two-layer structure develops. There is an inner region in which the temperature decreases from its maximum value at the plume axis to zero as ζ increases. In this region the vertical velocity remains at the same order of magnitude, and it decreases to zero only in a much wider outer region. The method of matched asymptotic expansions [10] is now used to find rational expansions for the temperature and vertical velocity fields valid for large σ.

The inner solution

The system of equations (2.9)–(2.13) is singular in the limit $\sigma \to \infty$. An inner scaling is required which is determined in part by the requirement that the

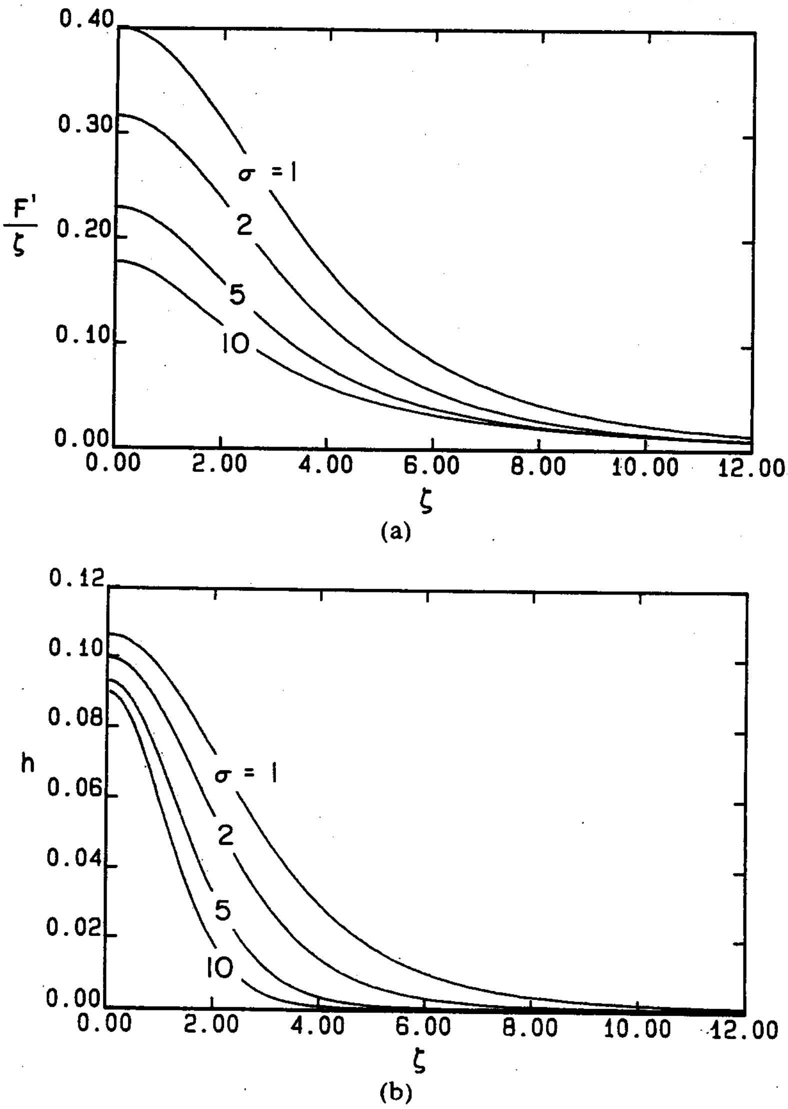

Figure 1. (a) The nondimensional vertical velocity F'/ζ, and (b) the nondimensional temperature h as functions of the similarity variable ζ for various values of the Prandtl number σ. Note the separation of scales as σ increases.

highest derivatives are retained. This ensures that the boundary conditions at the plume axis (2.12) can be applied. The conduction term is retained, and there is a balance between the two terms in the thermal energy equation, provided that

$$F(\zeta) = \sigma^{-1} f(\eta), \tag{3.1}$$

where

$$\eta = \epsilon^{-1}\zeta, \tag{3.2}$$

f and η are of order unity as $\sigma \to \infty$ in the inner region, and $\epsilon = \epsilon(\sigma) \ll 1$ is a small parameter which is, as yet, undetermined. The differential equations, scaled appropriately for the inner layer, are then

$$[\eta(f'/\eta)']' = -\epsilon^4\sigma\eta h - \sigma^{-1}f(f'/\eta)', \tag{3.3}$$

$$h' = -\frac{1}{\eta}fh, \tag{3.4}$$

$$2\pi\int_0^\infty f'h\,d\eta = 1. \tag{3.5}$$

The boundary conditions at the plume axis require

$$f = 0, \quad (f'/\eta)' = 0 \qquad (\eta = 0), \tag{3.6}$$

while matching with the outer solution determines the behavior of the inner solution as $\eta \to \infty$. Note that, since f is positive, Equation (3.4) shows $h \to 0$ as $\eta \to \infty$. We shall see that this is consistent with the matching requirement imposed by the outer solution. This also allows us to compute the constraint (3.5) from the inner solution only, since there is no contribution to the integral from the outer region.

From (3.3) we see that $\epsilon^4\sigma$ must be $O(1)$ in order to retain the highest derivative. In fact, it can be shown that $\epsilon^4\sigma$ must be $o(1)$, else $h \sim 0$ at leading order and the constraint (3.5) cannot be satisfied. There then exist asymptotic expansions of the form

$$f \sim f_0 + \epsilon^4\sigma f_1 + \cdots, \quad h \sim h_0 + \epsilon^4\sigma h_1 + \cdots \qquad (\sigma \to \infty). \tag{3.7}$$

The leading-order equations for the inner layer,

$$\left[\eta\left(\frac{f_0'}{\eta}\right)'\right]' = 0, \qquad h_0' = -\frac{1}{\eta}f_0h_0,$$

are readily integrated to yield

$$f_0 = \tfrac{1}{2}W_0\eta^2, \qquad h_0 = \left(\frac{1}{4\pi}\right)e^{-W_0\eta^2/4}. \tag{3.8a,b}$$

The leading-order centerline temperature $H_0 = 1/4\pi$ was determined from the integral constraint (3.5), but the leading-order centerline velocity W_0 must be determined by matching with the outer solution.

Equations governing the first-order corrections are

$$\left(\frac{f_1'}{\eta}\right)' = \frac{e^{-W_0\eta^2/4} - 1}{2\pi W_0\eta} \tag{3.9}$$

$$\left(h_1e^{W_0\eta^2/4}\right)' = -\frac{1}{4\pi}\frac{f_1}{\eta}. \tag{3.10}$$

These equations can be solved analytically, and the solutions will be given later [see Equations (3.19) and (3.20), and Appendix A] once the constants of integration have been determined. For now it suffices to note that as $\eta \to \infty$

$$f_1 \sim a\eta^2 \ln \eta + b\eta^2 + c, \tag{3.11}$$

where

$$a = \frac{-1}{4\pi W_0},$$

$$b = \frac{W_1}{2} + \frac{1 - \gamma + \ln(4/W_0)}{8\pi W_0},$$

$$c = -\frac{1}{2\pi W_0^2},$$

and γ is Euler's constant [see Appendix A, Equation (A4)]. The first-order correction to the centerline velocity W_1 must be determined by matching with the outer solution.

The outer solution and matching

In the outer region a balance between viscous dissipation and inertia is expected. Examination of Equation (2.9) shows that this requires F to be $O(1)$. Equation (2.10) then gives $h \sim 0$ at all orders in the outer region, which is consistent with the observation that $h \to 0$ as $\eta \to \infty$ in the inner region. Only the momentum equation remains, and this can be expressed as

$$[\xi(F'/\xi)']' = -F(F'/\xi)', \tag{3.12}$$

where $\xi = \delta^{-1}\zeta$ is the outer variable of order unity and $\delta(\sigma)$ is the outer scaling factor. The outer boundary condition

$$F'/\xi \to 0 \qquad (\xi \to \infty) \tag{3.13}$$

remains, while the behavior as $\xi \to 0$ is determined by matching with the inner solution. The inner limit of the outer solution is found by noting that F becomes small as $\xi \to 0$, so the right-hand side of (3.12) can be neglected and we obtain

$$F \sim A\xi^2 \ln \xi + B\xi^2 + C \qquad (\xi \to 0), \tag{3.14}$$

where A, B, and C are constants of integration.

Matching is achieved by requiring f to be equal to σF in an intermediate region defined by a general intermediate variable

$$\chi = (\delta/\epsilon)^{-\lambda}\eta = (\delta/\epsilon)^{1-\lambda}\xi \qquad (0 < \lambda < 1).$$

The expressions (3.11) and (3.14), rendered in terms of χ, are

$$\frac{1}{\sigma}f \sim \frac{1}{\sigma}\left(\frac{\delta}{\epsilon}\right)^{2\lambda}\tfrac{1}{2}W_0\chi^2 + \epsilon^4\left(\frac{\delta}{\epsilon}\right)^{2\lambda}\ln\left(\frac{\delta}{\epsilon}\right)a\chi^2$$

$$+\epsilon^4\left(\frac{\delta}{\epsilon}\right)^{2\lambda}(a\chi^2\ln\chi + b\chi^2) + \epsilon^4 c + \cdots$$

and

$$F \sim C - \left(\frac{\delta}{\epsilon}\right)^{-2(1-\lambda)}\ln\left(\frac{\delta}{\epsilon}\right)(1-\lambda)A\chi^2$$

$$+\left(\frac{\epsilon}{\delta}\right)^{-2(1-\lambda)}(A\chi^2\ln\chi + B\chi^2) + \cdots,$$

which are seen to be equal for all values of λ provided that

$$C = c\epsilon^4, \qquad A = a\epsilon^4(\delta/\epsilon)^2,$$

$$B = a\epsilon^4\left(\frac{\delta}{\epsilon}\right)^2\ln\left(\frac{\delta}{\epsilon}\right) + \tfrac{1}{2}W_0\sigma^{-1}\left(\frac{\delta}{\epsilon}\right)^2 + b\epsilon^4\left(\frac{\delta}{\epsilon}\right)^2.$$

Thus, (3.14) can be rewritten as

$$F \sim \sigma^{-1}\left(\frac{\delta}{\epsilon}\right)^2\tfrac{1}{2}W_0\xi^2 + \epsilon^4\left(\frac{\delta}{\epsilon}\right)^2\ln\left(\frac{\delta}{\epsilon}\right)a\xi^2$$

$$+\epsilon^4\left(\frac{\delta}{\epsilon}\right)^2(a\xi^2\ln\xi + b\xi^2) + O(\epsilon^4).$$

The first two terms of this expansion are both proportional to ξ^2. If either constituted the asymptotic form as $\xi \to 0$ of the leading-order, large-σ solution for F, then Equation (3.14) would imply $F \sim 0$ for all ξ. Therefore, these two terms must cancel to leave the third term as the leading-order term, which is required to be of order unity. These considerations lead to the proposals

$$\sigma^{-1} = \epsilon^4\ln(\delta/\epsilon), \qquad a = -\tfrac{1}{2}W_0, \qquad \epsilon^4(\delta/\epsilon)^2 = 1, \tag{3.15}$$

so that

$$F \sim a\xi^2\ln\xi + b\xi^2 \qquad (\xi \to 0). \tag{3.16}$$

The relations (3.15) are then readily manipulated to yield

$$W_0 = \frac{1}{\sqrt{2\pi}}, \qquad \delta = \epsilon^{-1}, \tag{3.17}$$

and

$$\epsilon^4 \ln \epsilon^{-2} = \sigma^{-1}. \tag{3.18}$$

The value of ϵ is given by the root of (3.18) that lies in the range $0 < \epsilon < e^{-1/4} \approx 0.779$, so that ϵ is a decreasing function of σ. All the scalings have now been determined, albeit implicitly, via the transcendental equation (3.18). The way is now clear for higher-order terms to be determined in a straightforward, if laborious, manner.

A point of interest is that the ratio of the vertical-velocity scales in the inner and outer layers is

$$\frac{W_{\text{outer}}}{W_{\text{inner}}} = \frac{\delta^{-2}}{\sigma^{-1}\epsilon^{-2}} = \left[\ln\left(\frac{1}{\epsilon^2}\right)\right]^{-1},$$

which is very much smaller than unity. This contrasts with the large-Prandtl-number solutions for two-dimensional convective boundary layers, in which the inner and outer velocities have the same order of magnitude [5]. This difference is almost certainly due to the cylindrical geometry, which means that the viscous stress generated in the inner layer has a much greater volume of outer fluid to raise for a given surface area at which the stress is applied.

We found W_0 analytically, but in order to determine W_1 it is necessary to solve the outer equation (3.12), and this is most easily done numerically. A shooting method, similar to that described in Section 2, was employed once the equation had been transformed using the coordinate stretching $\omega = \ln \xi$. Pertinent numerical results are shown in Table 2, and graphs of various terms in the expansions are displayed in Figures 2–4. The graphs for the inner solution (Figures 3 and 4) were plotted using analytical solutions which are derived in Appendix A, namely

$$f_1 = \tfrac{1}{2}W_1\eta^2 - \{1 - e^{-z} + z[E(z) - 1]\},$$

$$\frac{f_1'}{\eta} = W_1 - \frac{E(z)}{2\sqrt{2\pi}}, \tag{3.19}$$

$$h_1 e^z = H_1 - \frac{f_1 + z - E(z)}{8\pi}, \tag{3.20}$$

Table 2
Some Numerical Results for the Large-Prandtl-Number Solutions for an Axisymmetric Laminar Plume[a]

$a = -0.1995$	$W_0 = 0.3989$	$H_0 = 0.0796$
$b = 0.2730$	$W_1 = 0.0018$	$H_1 = 0.0122$
$c = -1.0$	$F_\infty = 3.267$	

[a]See text for definitions of the constants.

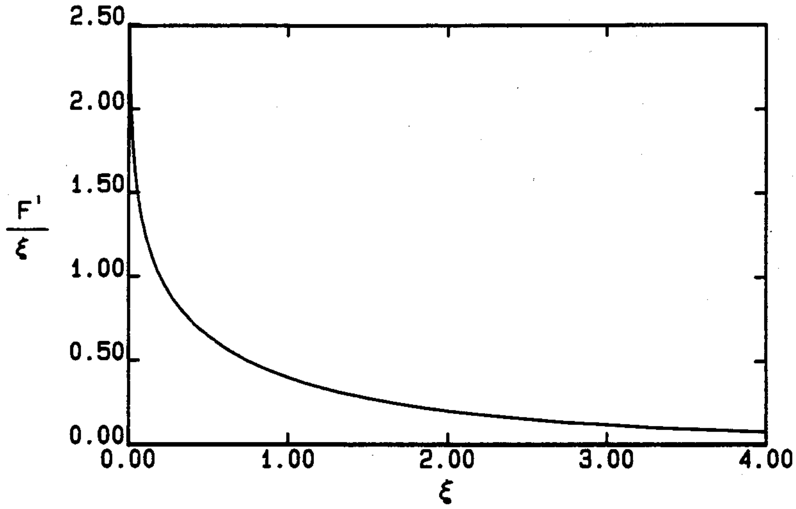

Figure 2. The nondimensional vertical velocity field F'/ξ in the outer region of the boundary layer as a function of the rescaled similarity variable ξ. Note the logarithmic singularity at $\xi = 0$ and the very slow (algebraic) decay as $\xi \to \infty$.

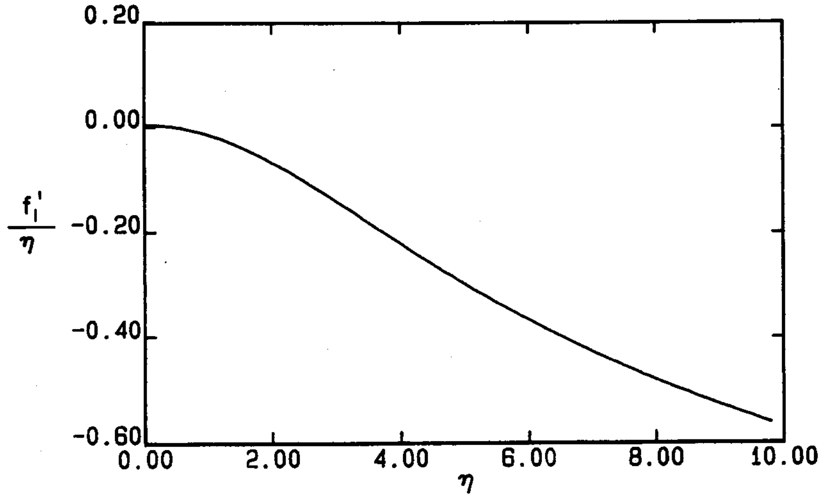

Figure 3. The first-order correction to the nondimensional vertical velocity f_1'/η in the inner region of the boundary layer. Note the negative logarithmic growth as $\eta \to \infty$, which matches to the outer solution as $\xi \to 0$.

where $z = \eta^2/4\sqrt{2\pi}$, $E(z) = E_1(z) + \gamma + \ln z$, and $E_1(z)$ is the first exponential integral [1].

4. Conclusions

Rational asymptotic expansions, in the limit $\sigma \to \infty$, have been found for a similarity solution of the axisymmetric boundary-layer equations which describe

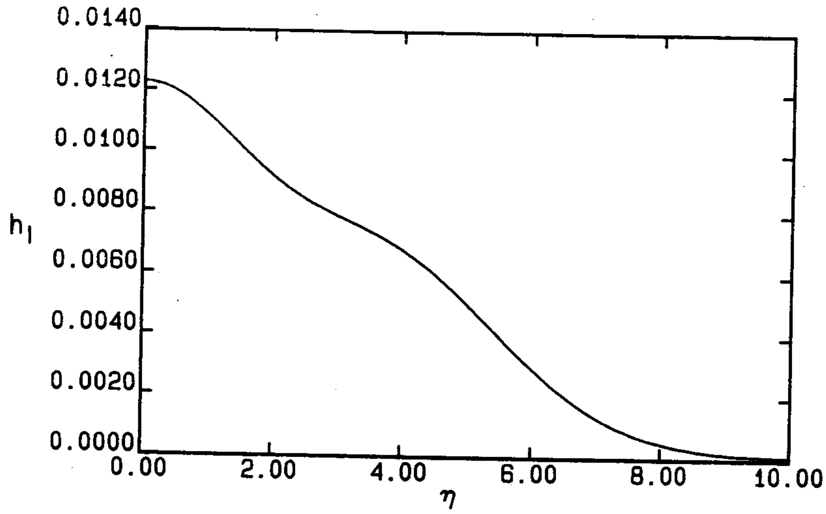

Figure 4. The first-order correction to the temperature field h_1 as a function of the rescaled similarity variable η.

a laminar, convective plume. The width of the inner, thermal boundary layer was shown to be of order $r \sim \epsilon G^{-1/4} z$, where G is the local, modified Grashof number defined by (2.8), and ϵ is that root of $\epsilon^4 \ln(1/\epsilon^2) = 1/\sigma$ that lies in the range $0 < \epsilon < e^{-1/4}$. There is an outer region, of width ϵ^{-2} times larger than the inner region, in which there is no buoyancy, and in which viscous dissipation balances inertia. The boundary-layer analysis is valid provided that the predicted width of the plume is much less than its height, so $\epsilon G^{1/4}$ must be much greater than unity. Since G is an increasing function of z, we see that the boundary-layer assumption is always correct sufficiently far above the buoyancy source. In a region of limited horizontal extent our analysis still applies provided the horizontal dimensions of the fluid region are at least of order $\epsilon^{-1} G_L^{-1/4}$ times its height L. However, it is clearly possible for $\epsilon^{-1} G_L^{1/4}$ to be much greater than unity while $\epsilon G_L^{1/4}$ is much less than unity. In this situation inertia is negligible everywhere, and there will be a narrow thermal boundary layer driving an outer Stokes flow. The analysis of Roberts [7] is then applicable.

A comparison of the computed results for $\sigma = 10$ with the predictions of the large-σ solutions is interesting. Let subscript a denote the asymptotic results; then, from the results in Table 1, it is straightforward to compute $F_\infty(10)/F_{\infty a} \approx 1.013$, $W(10)/W_a(10) \approx 1.249$, and $H(10)/H_a(10) \approx 1.005$, once $\epsilon^2(\sigma = 10) \approx 0.2805$ has been determined. So the low-order asymptotic expansions that have been presented are very accurate at predicting the mass flux and the centerline temperature even when σ is as small as 10, but the result for the centerline velocity is much less impressive.

The principal achievement of this paper has been the determination of the functional dependence on Prandtl number of the thermal and velocity fields in an axisymmetric laminar plume once the Prandtl number is large. This allows for confident scaling of more complex physical systems involving laminar plumes, and it shows the way for higher-order approximations to be obtained if they are required.

Appendix A

From Equation (3.9) we have, on integrating once,

$$f_1' = W_1\eta - \frac{\eta}{2\pi W_0}\int_0^\eta \left(1 - e^{-W_0\xi^2/4}\right)\frac{d\xi}{\xi}, \tag{A1}$$

where the constant of integration W_1 represents the first-order correction to the vertical velocity at the plume axis. Then, integrating by parts, we obtain

$$f_1 = \tfrac{1}{2}W_1\eta^2 - \frac{\eta^2}{4\pi W_0}\int_0^\eta \left(1 - e^{-W_0\xi^2/4}\right)\frac{d\xi}{\xi} + \frac{1}{4\pi W_0}\int_0^\eta \xi\left(1 - e^{-W_0\xi^2/4}\right)d\xi. \tag{A2}$$

Equations (A1) and (A2) can be rewritten in terms of exponential integrals as

$$\frac{f_1'}{\eta} = W_1 - \frac{1}{4\pi W_0}\left[E_1\left(\frac{W_0\eta^2}{4}\right) + \gamma + \ln\left(\frac{W_0\eta^2}{4}\right)\right], \tag{A3}$$

$$f_1 = \frac{-1}{4\pi W_0}\eta^2\ln\eta + \frac{1}{2}\left[W_1 + \frac{1}{4\pi W_0}\left(1 - \gamma + \ln\frac{4}{W_0}\right)\right]\eta^2 - \frac{1}{2\pi W_0^2}\left[1 - E_2\left(\frac{W_0\eta^2}{4}\right)\right], \tag{A4}$$

where $E_1(x) = \int_x^\infty (e^{-t}/t)\,dt$, $E_2(x) = e^{-x} - xE_1(x)$, and $\gamma = 0.57721\ldots$ is Euler's constant [1, p. 228 ff.].

Now, from Equations (3.5) and (3.7) we have at first order

$$0 = \int_0^\infty (f_0'h_1 + f_1'h_0)\,d\eta = \int_0^\infty \left[\left(h_1 e^{W_0\eta^2/4}\right)W_0\eta e^{-W_0\eta^2/4} + f_1'h_0\right]d\eta.$$

Integration by parts gives

$$2H_1 = \frac{1}{4\pi}\int_0^\infty \left(\frac{2f_1}{\eta} - f_1'\right)e^{-W_0\eta^2/4}\,d\eta. \tag{A5}$$

Note, from (A1) and (A2) that

$$\frac{2f_1}{\eta} - f_1' = \frac{1}{4\pi W_0}\left[\eta - \frac{4}{W_0\eta}\left(1 - e^{-W_0\eta^2/4}\right)\right], \tag{A6}$$

so (A5) can be evaluated to yield

$$H_1 = \frac{1}{16\pi^2 W_0^2}(1-\ln 2). \tag{A7}$$

Finally, from (3.10) we have

$$\left(h_1 e^{W_0\eta^2/4}\right)' = -\frac{1}{8\pi}\left[\left(\frac{2f_1}{\eta} - f_1'\right) + f_1'\right],$$

Equation (A6) can again be used, and one integration gives

$$h_1 e^{W_0\eta^2/4} = H_1 - \frac{1}{8\pi}f_1 + \frac{1}{16\pi^2 W_0^2}\left[E_1\left(\frac{W_0\eta^2}{4}\right) + \gamma + \ln\left(\frac{W_0\eta^2}{4}\right) - \frac{W_0\eta^2}{4}\right]. \tag{A8}$$

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