NOTES AND CORRESPONDENCE

A Note on Baroclinic Instability of the Zonal Wind to Short Waves[1]

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The mathematical description of baroclinic instability of the zonal wind yields the singular Sturm-Liouville system (Miles, 1964a)

$$(P\psi')'+\lambda(U-c)^{-1}\psi-\alpha^2\psi=0, \tag{1}$$

$$(U-c)\psi'-(U'-\kappa c)\psi=0 \quad \text{at} \quad z=0, \tag{2}$$

$$P\psi\psi'=0 \quad \text{at} \quad z=1, \tag{3}$$

for the potential $\psi(z)$; $P(z)$, $U(z)$, and $\lambda(z)$ are prescribed functions of the independent variable z; α and κ are prescribed, positive-real parameters; and the admissible values of the dimensionless wave speed c are to be determined (complex c implies instability). We consider here the asymptotic behavior of c as $\alpha\rightarrow\infty$. Our results do not differ significantly from those given previously by Green (1960) without derivation and by Miles (1964a) on the basis of a heuristic derivation, but our analysis places these results on a reasonably firm mathematical foundation.

We assume (as is physically justifiable) that: $P(z)$ is analytic in a domain D, to be defined below, with

$$P(0)=1, \quad P(z)\sim P_1(1-z)^2 \quad \text{as} \quad z\rightarrow 1, \tag{4a,b}$$

where P_1 is a constant; $U(z)$ is monotonic increasing over $z=[0,1]$, and $U-c$ is analytic in D, with a simple zero at z_c $[U(z_c)=c]$; and λ is given by

$$\lambda=\beta-(PU')', \tag{5}$$

where β is a positive-real constant. Remarking that the exponents of (1) at the singularity $z=1$ are $\frac{1}{2}\pm k$, where

$$k=\{P_1^{-1}[\alpha^2-\beta(U_1-c)^{-1}]+\tfrac{1}{4}\}^{\frac{1}{2}}, \quad \Re(k)>0, \tag{6}$$

and the subscript 1 implies evaluation at $z=1$, we define new variables ϕ and w by

$$\phi(w)=w^{-k}P^{\frac{1}{4}}\psi, \tag{7}$$

and

$$w(z)=\exp\left(-P_1^{\frac{1}{2}}\int_{z_c}^{z}P^{-\frac{1}{2}}dz\right). \tag{8}$$

Substituting (7) and (8) into (1), we obtain

$$w\phi''(w)+(2k+1)\phi'(w)+(1-w)^{-1}q(w)\phi(w)=0, \tag{9}$$

where

$$q(w)=w^{-1}(1-w)\{\tfrac{1}{4}+P_1^{-1}[\lambda(U-c)^{-1}-\beta(U_1-c)^{-1}] -\tfrac{1}{4}P^{\frac{3}{2}}P_1^{-1}(P^{-\frac{3}{2}}P')'\}. \tag{10}$$

We note that $q(0)$ is finite.

The form of (9) suggests the use of the hypergeometric differential equation as a comparison equation for the determination of ϕ. Accordingly, we define the function $G(w)$ by

$$\phi=F(w)G(w), \quad F={}_2F_1(a,b;a+b+1;w), \tag{11a,b}$$

where

$$a,\, b=k\pm(k^2+\nu)^{\frac{1}{2}}, \quad \nu=q(1)=P_1^{-\frac{1}{2}}(\lambda_c/U_c'), \tag{12a,b}$$

and the w-plane is cut along the real axis, from 1 to ∞. Substituting (11) into (9) and invoking (3), (4) and (8), we find that G satisfies the integral equation

$$G(w)=1+\int_0^w v^{-2k-1}F^{-2}(v)dv\int_0^v(u-1)^{-1} \times[q(u)-q(1)]u^{2k}F^2(u)G(u)du. \tag{13}$$

We now define D as the smallest domain in the z-plane such that $|(w-1)^{-1}[q(w)-q(1)]|$ is uniformly bounded in a region R of the w-plane that includes the segment

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$[0,w_0]$, where $w(0)=w_0$, and the point $w=1$. Invoking standard results for the hypergeometric function in (13), we can prove that

$$G(w)=1+0(k^{-1}) \quad (14)$$

uniformly in R, provided that w_0 is bounded uniformly with respect to k. We shall consider $w_0\rightarrow\infty$, which implies $z_c\rightarrow 1$, separately.

Substituting the eigen solution $\psi=w^kP^{-\frac{1}{4}}FG$ into the boundary condition (2), we find that

$$(U_0'-\kappa c)/(c-U_0)=kP_1^{\frac{1}{2}}+\tfrac{1}{4}P'(0)+w_0P_1^{\frac{1}{2}} \times\{[F'(w_0)/F(w_0)]-[G'(w_0)/G(w_0)]\}. \quad (15)$$

If w_0 is bounded uniformly with respect to k, (15) can be satisfied only if

$$|U_0-c|=0(k^{-1}). \quad (16)$$

Assuming that $c=c_r+ic_i$ may be expressed as a power series in inverse powers of k and equating like powers of k in (15), we obtain

$$c=U_0+(U_0'-\kappa U_0)k^{-1}P_1^{-\frac{1}{2}}+0(k^{-2}), \quad (17)$$

$$c_i=-(U_0'-\kappa U_0)|k|^{-2}P_1^{-\frac{1}{2}} \times\lim_{k\to\infty}\{\mathrm{Im}[w_0F'(w_0)/F(w_0)]\}+0(|k|^{-3}). \quad (18)$$

Comparing (17) with the Taylor series about $z=0$ for $U(z)$, evaluated at z_c, we can obtain the asymptotic expansion of z_c with respect to k. Calculating w_0 from this expansion and invoking standard relationships for the hypergeometric function in (18), we obtain

$$c_i=\pi\lambda_0\alpha^{-2}[1-\kappa(U_0/U_0')]\exp\{-2[1-\kappa(U_0/U_0')]\}+0(\alpha^{-3}), \quad (19)$$

and

$$c=U_0+(U_0'-\kappa U_0)\alpha^{-1}+0(\alpha^{-2}). \quad (20)$$

We note that (19) and (20), after neglecting κU_0 with respect to U_0', are identical with the results obtained heuristically by Miles [1964a, (13.11) and (13.14b)] after continuing a Green-Liouville solution through a singularity of exponents zero and one.

It remains to discuss the solution as $z_c\rightarrow 1$. A rigorous analysis in this case would appear to be extremely difficult. The difficulty lies primarily in the fact that (1) describes a differential equation that has only regular singularities in D if z_c is bounded away from 1, but has an irregular singularity at $z=1$ in the limit $z_c\rightarrow 1$. We also remark that k, as defined in (6), is not necessarily of order α, and that the transformation (8) is not applicable, for $z_c\rightarrow 1$.

Letting $P(z)=(1-z)^2$, we consider approximating (1) by

$$[(1-z)^2\psi']'+[(\beta/U_1')(z-z_c)^{-1}-\alpha^2]\psi=0, \quad (21)$$

which can be transformed to the hypergeometric equation. Miles (1964b) has solved an equation equivalent to (21) with the boundary conditions (2) and (3) and demonstrated that there is one, and only one, eigenvalue and that this eigenvalue is complex. We therefore are led to conjecture that the one (complex) eigenvalue already found for the general problem is the only possible eigenvalue and that no further eigenvalue exists in the region $z_c\rightarrow 1$. Unfortunately, we have not been able to demonstrate that the error implied by the approximation of (1) by (21) is uniformly small.

REFERENCES

Green, J. S. A., 1960: A problem in baroclinic stability. *Quart. J. R. Meteor. Soc.*, **86**, 237–251.

Miles, J. W., 1964a: Baroclinic instability of the zonal wind. *Rev. of Geophysics*, **2**, 155–176.

——, 1964b: Baroclinic instability of the zonal wind: part III. *J. Atmos. Sci.*, **21**, 603–609.