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A Note on the Howard–Malkus–Whitehead Floating Heat Sources

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Abstract—Some deficiencies in a recent paper by Howard, Malkus and Whitehead are examined. The problem is reformulated in terms of an integro-differential equation, from which both asymptotic and numerical solutions are obtained.

1. Introduction

In a recent publication Howard, Malkus and Whitehead (1970) (herein referred to as HMW, followed by the appropriate equation number) consider the motion of two infinite, line heat sources in a viscous fluid of infinitely large Prandtl number. The sources are constrained to move at a fixed depth between two stress-free, horizontal planes held at zero temperature. The ensuing velocities are considered to be sufficiently small that the equations can be linearized. The model is motivated by a desire to understand the mechanism involved in the plate-like motion of the earth's crust as investigated recently by McKenzie and Parker (1967), Isacks, Oliver and Sykes (1968), McKenzie (1969) and others.

HMW deduce that if the Peclet number, R, which is proportional to the thermal source strength per unit length, is larger than a critical value R_c, the sources attain a constant, non-zero velocity after a sufficiently large time. What happens if the Peclet number is less than R_c is not considered. HMW performed some experiments which did not altogether confirm their conclusions. The experimental velocities were constant for some time, but subsequently became zero.

With examination, the arguments upon which their conclusions are based are open to objections. An expansion [HMW (1.29)] is made which is actually an *a priori* assumption of constant velocity. How this affects the final answer is not clear. Further, the analysis leads to an equation for the final velocity, U (HMW (1.29); $b = \infty$ therein). This equation has not only the cited, non-zero solution, but also the solution $U = 0$. In view of the experimental result one wonders if $U = 0$ is not the correct solution. Unfortunately, the approach used by HMW does not allow one to decide between the two possible alternatives.

We reformulate the problem in terms of an integro-differential equation, from which it is possible to determine analytically the solution for both small and large times, for all Peclet numbers. The integro-differential equation is ideally suited for numerical integration, and solution curves for two Peclet numbers and a variety of initial displacements are presented. From these calculations, we find that the final velocity is indeed non-zero, constant and given by HMW's formula (1.30) if $R > R_c$, while if $R < R_c$, the sources, after the initial motion, continue to separate with an ever decreasing velocity.

2. Analysis

The problem in non-dimensional form is shown in Fig. 1, which is assumed symmetric about $x = 0$. We seek $a(t)$ given $a(0)$, d and R, where: α is the coefficient of expansion; g the acceleration due to gravity; Q the strength of the heat source per unit length situated at $[a(t), -d]$; h the distance between the upper and lower boundaries;

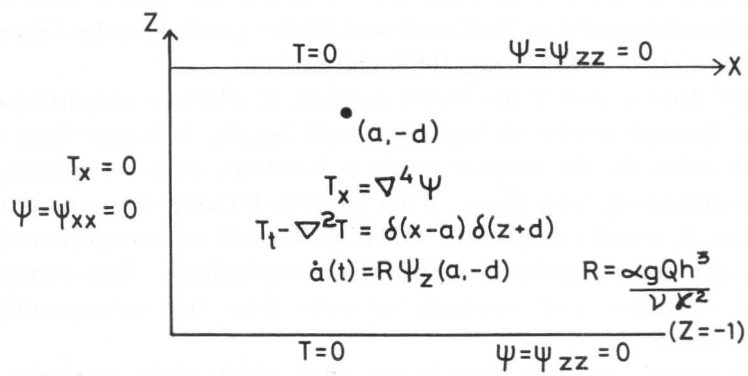

Figure 1. The geometry, equations and boundary conditions.

ν the kinematic viscosity; and κ the thermometric conductivity. A detailed explanation of the non-dimensionalization and the assumptions involved is contained in HMW.

Introducing the Fourier transforms

$$\theta(k, m) = \int_0^\infty \cos kx \, dx \int_{-1}^0 T(x, z) \sin m\pi z \, dz \tag{1a}$$

and

$$\phi(k, m) = \int_0^\infty \sin kx \, dx \int_{-1}^0 \psi(x, z) \sin m\pi z \, dz, \tag{1b}$$

solving the resulting equations, carrying out the Fourier inversion for ψ, differentiating with respect to z and setting $x = a(t)$, $z = -d$, we obtain

$$\dot{a}(t) = R \sum_{m=1}^{\infty} m \sin(2\pi m d) \int_0^\infty k(k^2 + \pi^2 m^2)^{-2} \, dk \cdot \int_0^t \exp[-(k^2 + \pi^2 m^2)\eta] F(t, \tau) \, d\eta, \tag{2}$$

where

$$\eta = t - \tau \tag{3a}$$

and

$$F(t, \tau) = \sin\{k[a(t) + a(\tau)]\} + \sin\{k[a(t) - a(\tau)]\}. \tag{3b}$$

A very good approximation is obtained by considering only the $m = 1$ term in the right-hand-side of (2); HMW state that for $d = \frac{1}{4}$, the first term is typically 98% of the total sum. Making this approximation and interchanging the order of integration, we obtain the integro-differential equation for $a(t)$

$$\dot{a}(t) = R' \int_0^t \{G[a(t) + a(\tau), \eta] + G[a(t) - a(\tau), \eta]\} \, d\eta, \tag{4}$$

where

$$G(x, \eta) = \int_0^\infty k(k^2 + \pi^2)^{-2} \exp[-(k^2 + \pi^2)\eta] \sin kx \, dk \tag{5a}$$

$$= \tfrac{1}{8}[(x - 2\pi\eta) \exp(-\pi x) \mathrm{Erfc}(\pi\eta^{1/2} - \tfrac{1}{2}x\eta^{-1/2}) + (x + 2\pi\eta) \exp(\pi x) \mathrm{Erfc}(\pi\eta^{1/2} + \tfrac{1}{2}x\eta^{-1/2})] \tag{5b}$$

and

$$R' = R \sin 2\pi d. \tag{6}$$

From (4) and (5) we immediately obtain

$$\dot{a}(t) = \tfrac{1}{2}R'[a(0) - \tfrac{1}{2}\pi t]t \exp[-2\pi a(0)][1 + O(t^2)] \qquad (t \to 0) \qquad (7)$$

and hence

$$a(t) = a(0) + \tfrac{1}{4}R'[a(0) - \tfrac{1}{3}\pi t]t^2 \exp[-2\pi a(0)][1 + O(t^2)] \qquad (t \to 0). \qquad (8)$$

Considering large t, since $G(x, \eta)$ is an exponentially decreasing function of η, we use the concepts employed in Laplace's method of evaluating the dominant term in the asymptotic expansion of certain integrals to write

$$\dot{a}(t) \sim R' \int_0^\infty \{G[2a(t), \eta] + G[\dot{a}(t)\eta, \eta]\} \, d\eta \qquad (t \to \infty). \qquad (9)$$

Using the integral representation (5a) and interchanging the order of integration, we evaluate the first integral in the right-hand-side of (9) as

$$\frac{1}{8\pi^2} R' a(t)[1 + 2\pi a(t)] \exp[-2\pi a(t)] \qquad (10)$$

and the second integral as

$$\frac{R'}{\dot{a}(t)} \left[\frac{1}{4\pi^2} - \frac{1}{2\pi \sqrt{\dot{a}^2(t) + 4\pi^2}} \right]. \qquad (11)$$

If $\dot{a}(t)$ tends to zero as t tends to infinity, the dominant term of (11) becomes

$$\frac{R'}{32\pi^4} \dot{a}(t). \qquad (12)$$

Substituting (10) and (12) into (9), we obtain

$$\dot{a}(t) \sim \frac{R'}{8\pi^2} \left(1 - \frac{R'}{32\pi^4}\right)^{-1} a(t)[1 + 2\pi a(t)] \exp[-2\pi a(t)]$$
$$[\dot{a}(t) \to 0, \; t \to \infty], \qquad (13)$$

which has the solution

$$t - t_0 = \frac{32\pi^4 - R'}{4\pi^2 R'} \{Ei[2\pi a(t)] - Ei[2\pi a(t_0)]$$
$$- e^{-1} Ei[2\pi a(t) + 1] + e^{-1} Ei[2\pi a(t_0) + 1]\}. \qquad (14)$$

This is the large-time solution for $R' < 32\pi^4 \equiv R_c$, the critical Peclet number as determined by HMW. It would clearly be convenient if t_0 could be set equal to zero in (14). However, the approximation (9) is invalid for small t, and (14) with $t_0 = 0$ does not compare well with the numerical solutions obtained below. An

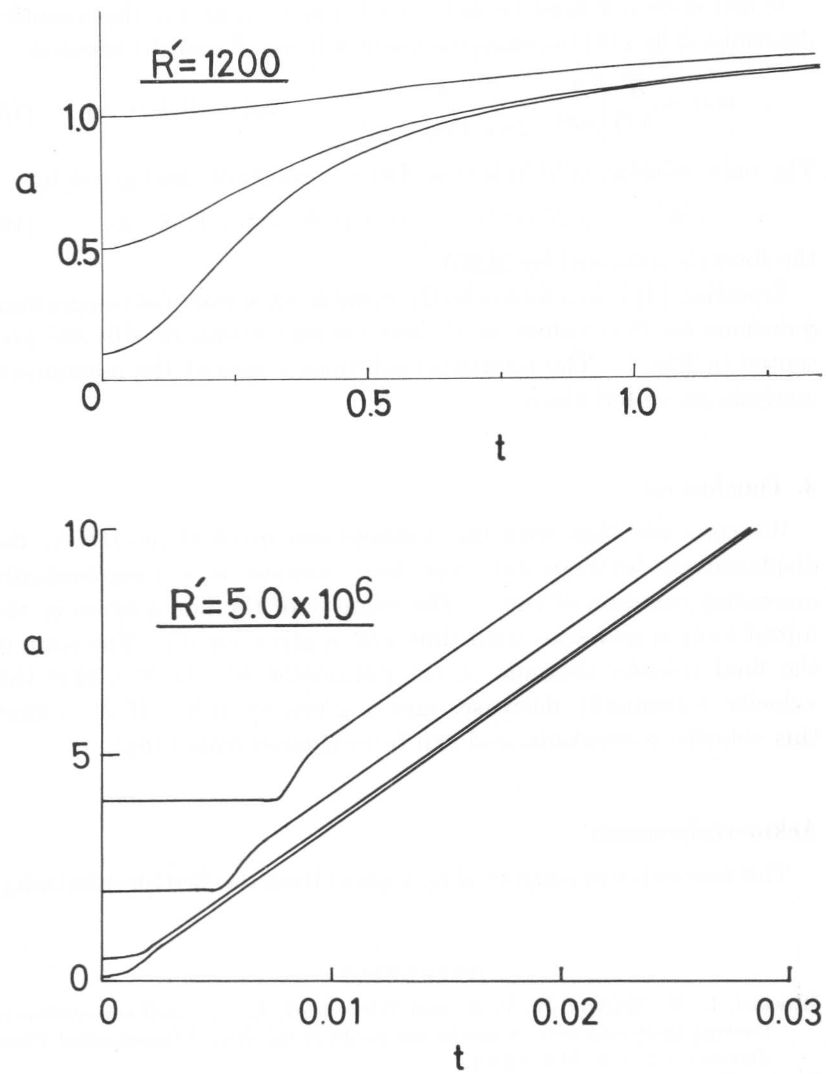

Figure 2. The displacement as a function of time, as obtained by numerically solving the integral Eq. (4).

appropriate value of t_0 can be obtained from (8), from which $a(t_0)$ can be calculated.

If $\dot{a}(t)$ does not tend to zero as t tends to infinity, the quantity determined by (10) becomes exponentially small and (9) becomes

$$\dot{a}(t) \sim \frac{R'}{\dot{a}(t)}\left[\frac{1}{4\pi^2} - \frac{1}{2\pi\sqrt{\dot{a}^2(t)+4\pi^2}}\right] \qquad [\dot{a}(t) \nrightarrow 0,\ t \to \infty]. \qquad (15)$$

The only solution of (15) is that $\dot{a}(t)$ is a constant, and given by

$$\dot{a}^2(t) = \pi^2\{[(R'/4\pi^4)+1]^{1/2}+1\}\{[(R'/4\pi^4)+1]^{1/2}-3\}, \qquad (16)$$

the formula obtained by HMW.

Equation (4) is in a form ideally suitable for numerical integration. Solutions for two values of R' and various values of $a(0)$ are presented in Fig. 2. The numerical solutions augment the asymptotic analysis presented above.

3. Conclusions

We conclude that with the assumptions invoked by HMW, the displacement between two line heat sources is a monotonically increasing function of time. The velocity of the line sources in the initial motion increases with time and is given by (7). The form of the final velocity depends on the parameter R'. If $R' < 32\pi^4$ this velocity continually decreases and is given by (13). If $R' > 32\pi^4$, this velocity is constant, and can be evaluated from (16).

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