# On the stability of a series of double-diffusive layers

HERBERT E. HUPPERT*

**Abstract**—An analytic expression is presented fitting Turner's experimental results on the flux through a diffusive interface in a heat/salt system. This expression is then used in an explicit calculation of the positions of equilibrium and the stability of a system consisting of one intermediate layer between two semi-infinite layers of constant properties. If both interfaces are in the regime for which the nondimensional ratio of the salt flux to the heat flux is independent of the density ratio $\beta \Delta S / \alpha \Delta T$, a family of equilibrium positions exists and is neutrally stable. If either interface is not in this regime the system is unstable. The criteria for stability are obtained for general flux laws and it is shown that these are also the criteria for any number of intermediate layers. A simple re-interpretation of these criteria is presented which renders them valid also for fingering interfaces. The salinity field is shown to be the destabilizing agent for diffusive interfaces, and the temperature field the destabilizing agent for fingering interfaces. Relevant oceanographic observations are presented and discussed.

## INTRODUCTION

THE CONCEPT of a series of vigorously convecting layers separated by thin interfaces has recently been introduced and accepted in oceanography. Various observations under quite different conditions have been made of this microstructure. The layers are regions of remarkably uniform temperature and salinity. The interfaces are very much thinner than the layers and are regions of comparatively large gradients of temperature and salinity. Laboratory experiments simulating such conditions have also been undertaken and have led to an increased understanding of the phenomenon.

The formation of layers is a consequence of the stratification with respect to the two different components: heat and salt, the diffusivities of which are different. A fluid in which temperature and salinity decrease with depth forms tall, thin convection cells, called 'fingers'. In these cells, descending elements diffuse their heat horizontally more rapidly than their salinity. They thereby become heavier than their surroundings and continue to descend. A similar mechanism pertains to the ascending elements. [A striking photograph of this phenomenon is to be found in TURNER (1967)]. STERN (1969) shows that the fingers are themselves unstable when the salt flux exceeds a critical value. He suggests that this instability restricts the fingers to thin interfaces, separating turbulently convecting regions. Alternatively, in a fluid in which both temperature and salinity increase with depth, buoyant elements use the potential energy of the temperature field to rise against the stabilizing effect of the salt. Elements can only rise a finite height before losing their positive buoyancy. Linearised theory, as originally discussed by VERONIS (1965) suggests that these elements now descend and the convection takes place as an oscillatory mode, as has been documented by SHIRTCLIFFE (1967). When non-linear effects are significant, however, the convection forms uniform regions bounded by interfaces across which

*Department of Applied Mathematics and Theoretical Physics, Cambridge, England.

there are substantial diffusive fluxes of both heat and salt. Almost all geophysical observations fall within the non-linear range. In all these forms of convection viscosity inhibits motions of sufficiently small amplitude.

These types of convection are not restricted to heat/salt systems. STERN and TURNER (1969) showed experimentally that they can be produced in any two-component systems; in particular in a solution of KCl and NaCl, for which the ratio of the diffusivities is 1·3 compared with 100 for heat and salt. For this reason the term *double-diffusive convection* has recently been given as a generic name for any form of convection involving two components of different diffusivities.

As a first step in detailing the transfer of heat and salt across a series of layers and interfaces, TURNER (1965, 1967) carried out a number of quantitative experiments with a *single* interface. He measured the heat and salt fluxes for a wide range of density differences across both types of interface: the finger interface, for which the destabilizing component (salt) is in excess above the interface; and the diffusive interface, for which the destabilizing component (heat) is in excess below. His results indicate that for temperature and salinity differences across the interfaces of magnitude $\Delta T$ and $\Delta S$, the heat and salt flux vary systematically with the density ratio $\beta \Delta S / \alpha \Delta T$, where $\alpha$ and $\beta$ are the proportional density changes due to unit changes in temperature and salt.

We employ these results to consider a *series* of layers and address ourselves here to determining under what conditions such a series is stable. Are there circumstances for which the interfacial flux, driven by the density difference across the interface, causes that density difference to become zero, thus making the interface collapse and the two adjoining layers merge as one? Or, can the destabilizing component equilibrate across the interface, thereby halting the convection process? Can either of these phenomena continue *via* adjoining interfaces, replacing a system of layers by a more homogeneous situation?

Using flux laws for diffusive interfaces determined from TURNER's (1965) experiments, we investigate the equilibrium positions and delineate the conditions for stability of one intermediate layer between two semi-infinite ones. In the following sections we show how the results obtained can be simply extended for: general flux laws; many intermediate layers; and either diffusive or fingering interfaces. A summary of the results obtained is presented in the concluding section with a discussion of the relevance of the theory to interpreting oceanographic data.

## FLUX LAWS FOR A DIFFUSIVE INTERFACE

The stability conditions for a series of layers will be presented for rather general flux laws in a later section. Much insight is gained, however, by first considering an explicit example — the diffusive interface. Quite a number of investigations have attempted to determine the equations governing the heat and salt transfer by purely theoretical considerations; not one has yet been satisfactory. Almost all require some *ad hoc* hypotheses and even then do not fit the available data. The approach used here is quite different. From the investigations of TURNER (1965) we can determine *experimental* flux laws. We find that a simple expression fits his data exceedingly well.

Consider two turbulently convecting layers of depth $h$, mean temperatures $T_1$ and $T_2 (T_2 - T_1 = \Delta T > 0)$, and zero salinity, separated by a solid plane. At high Rayleigh

numbers the motion within each layer takes the form of a very nearly isothermal central section and thin boundary layers at the boundaries. The structure of these boundary layers, their thickness insensitive to the layer depths, determines the heat flux across the plane, say $H_{SP}$. Such a motion implies that the heat flux is independent of the layer depths and is given by

$$Nu \equiv H_{SP}(k_T \Delta T/h)^{-1} = cRa^{1/3} \equiv c(g\alpha\Delta Th^3/k_T v)^{1/3}, \qquad (1)$$

where: $Nu$ is the Nusselt number, the heat flux divided by the purely conductive value; $Ra$ the Rayleigh number; $g$ the acceleration due to gravity; $k_T$ the thermal diffusivity; and $c$ a constant, determined either theoretically or experimentally. Equation (1) is still slightly controversial; more comprehensive discussions with summaries of theoretical and experimental investigations are contained in HOWARD (1966), SPIEGEL (1967) and ROSSBY (1966, 1969).

Generalising the argument for two layers of mean salinities $S_1$ and $S_2$ ($S_2 - S_1 = \Delta S > 0$) separated by a thin interface, TURNER (1965) obtains on dimensional grounds the formula

$$Nu = F(\beta\Delta S/\alpha\Delta T)Ra^{1/3}, \qquad (2)$$

where $F$ incorporates the dependence of some physical properties which are constant for given components — molecular diffusivities for example. Equation (2) implies that the ratio of the heat flux through the interface, say $H$, to that across a solid plane, determined from (1), is given by

$$H/H_{SP} = c^{-1}F(\beta\Delta S/\alpha\Delta T) \equiv G(\beta\Delta S/\alpha\Delta T). \qquad (3)$$

Taking $c$ as 0·085, we plot TURNER's experimental points in Fig. 1.

Through these points is drawn the curve

$$G(x) = 3\cdot8\ x^{-2}. \qquad (4)$$

Consider the function $\mathscr{G}(x) = G(x)/(3\cdot8x^{-2})$ elevated at each of the 63 data points. This array has unit mean and a standard deviation of 0·21. We compare this figure with the cited error of 30% for the experiments to conclude that equation (4) fits the
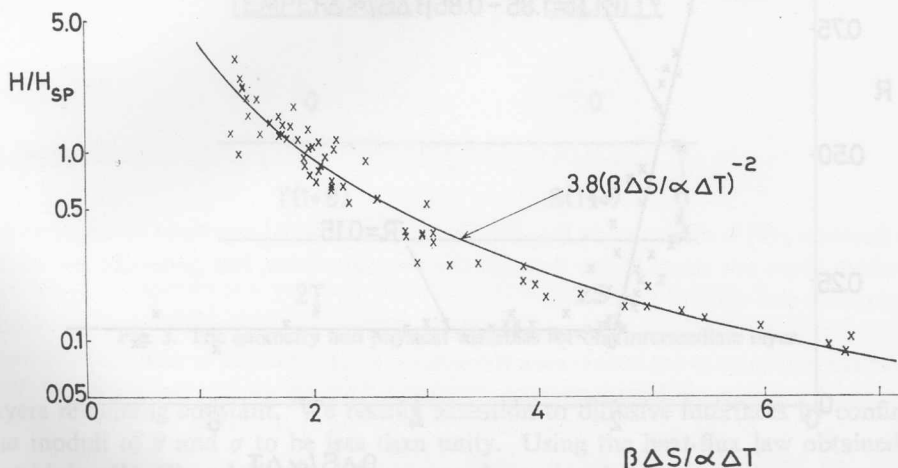


Fig. 1. The ratio of the heat flux across a diffusive interface to that across a solid plane. Three experimental points, (1·2, 9·5), (1·5, 8·5), and (1·5, 9·3), whose validity is in doubt (Turner; private communication) have been omitted.

data to within the experimental error. A sharper test, insensitive to the size of the experimental error, is obtained by evaluating

$$1\cdot02\,(1-0\cdot008x)$$

as the linear, least squares fit to $\mathscr{G}$ over the experimental range of $1\cdot3 \leqslant x \leqslant 6\cdot8$. The magnitude of $0\cdot008$ compared with 1 and the standard deviation test above leads us to assert that the function (4) fits the data exceedingly well. We propose this function as one towards which future theoretical consideration should be directed.

An interesting feature of equation (4) is the value $3\cdot8$ of $G(1)$. TURNER (1965) notes that in the neighbourhood of $x = 1$, $G(x) > 1$ (the heat flux is greater than that across a solid plane). He suggests that this is attributable to the distortion of the interface, which increases the surface area, and thus leads to a larger heat transfer. We propose a different, complementary explanation. HERRING (1963, 1964) considers three-dimensional thermal convection between parallel plates. He numerically solves the mean field equations after making the weak coupling approximation in which the self-interactions of the fluctuating components are neglected. He considers both free boundaries (zero stress) and rigid boundaries (zero velocity). At large Rayleigh numbers equation (1) is obtained, with the heat flux for free boundaries determined as $2\cdot3$ times larger than that for rigid boundaries. This is due to the weaker constraint of free boundaries which allows fluid to flow along the boundaries, transfer heat, and then complete the convection cycle. Such a process will obviously be more efficient at transferring heat than the rigid boundary equivalent. We propose that a similar mechanism is operative in heat/salt convection across a diffusive interface, as reflected in the values of $G$ larger than 1.

The most remarkable, and best known facet of TURNER's results is the functional
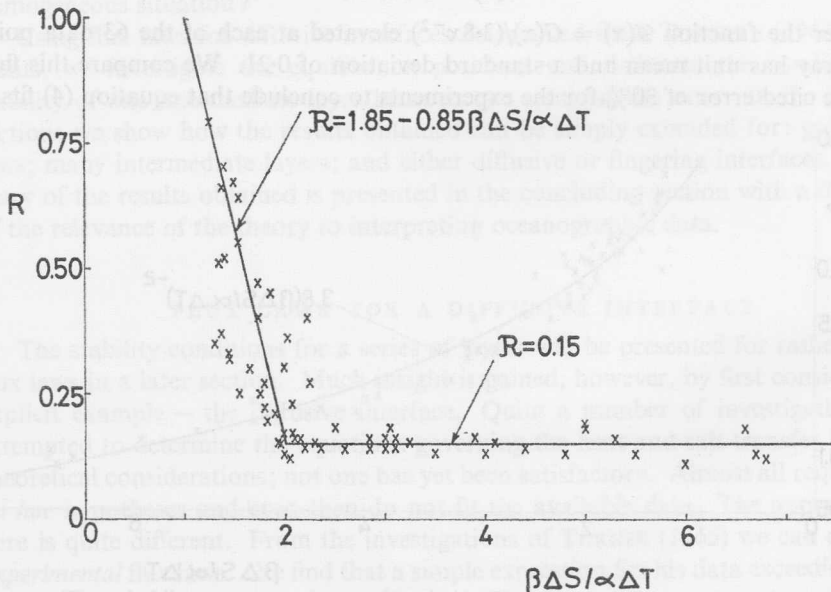


Fig. 2. The ratio of the potential energy gained by the salt to that released by the heat. Three points, $(1\cdot2, 0\cdot79)$, $(1\cdot5, 0\cdot8)$, and $(1\cdot5, 1\cdot7)$ have been omitted, as explained in the caption to Fig. 1.

form taken by the salt flux, $F_S$, a non-dimensional representation of which is $\beta F_S/\alpha H \equiv R$. The quantity $R$ is the ratio of the potential energy gained by the elevation of salt above the interface to that released by the accompanying transfer of heat. TURNER's measurements of this ratio and his suggested best-fit curve are reproduced in Fig. 2. For values of $\beta\Delta S/\alpha\Delta T$ greater than two, $R$ is a constant, 0·15, to within experimental accuracy. As $\beta\Delta S/\alpha\Delta T$ decreases from two the value of $R$ increases, approaching unity as the density difference across the interface approaches zero, where the experiments indicate that heat and salt are transported by the same turbulent eddies. We differentiate between these two regimes by calling the former the *constant* regime, the latter the *variable* regime. The occurrence of these two regimes is not limited to diffusive interfaces; the existence of a constant regime has been verified for fingering interfaces (TURNER, 1967; STERN and TURNER, 1969) and a variable regime must also exist.

The flux of density, $F_\rho = \alpha H - \beta F_S$, is most conveniently non-dimensionalised with respect to $\alpha H_{SP}$. The quantity $\mathscr{F} = F_\rho/\alpha H_{SP}$: is never negative, because $\alpha H \geqslant \beta F_S$; is zero for $\beta\Delta S/\alpha\Delta T = 1$, where $\alpha H = \beta F_S$; and tends to zero as $\beta\Delta S/\alpha\Delta T$ tends to infinity. Hence, somewhere in the range $1 < \beta\Delta S/\alpha\Delta T < \infty$ it is a maximum. Evaluating $\mathscr{F}$ using (3), (4) and the functional form for $R$ presented in Fig. 2, we conclude that the maximum occurs at $\beta\Delta S/\alpha\Delta T = 2$, exactly coincident with the intersection of the variable and constant regimes. If (4) were replaced by $G(x) \propto x^{-r}$, the maximum would occur at $x = 2$ if $r < 2$, and at $x = r/(r-1)$ if $r > 2$.

## THE STABILITY OF ONE INTERMEDIATE LAYER

### Specific flux laws

In this section we consider the stability of the simple, yet illuminating system of one intermediate layer between two infinitely deep layers. The temperature and salinity of the layers are as depicted in Fig. 3, the properties of the two infinitely deep
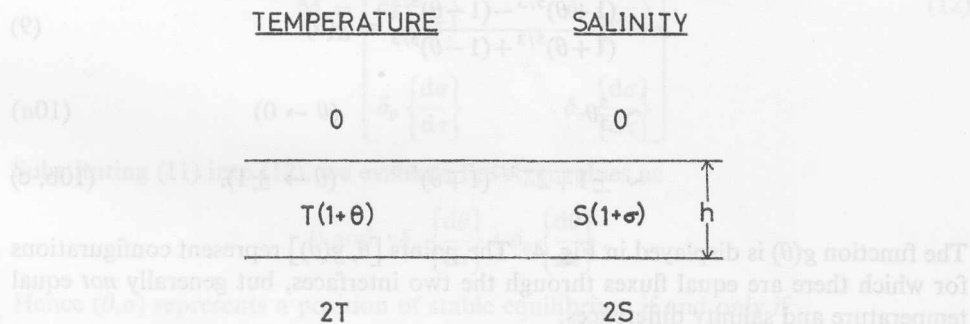


Fig. 3. The geometry and physical variables for one intermediate layer.

layers remaining constant. We restrict attention to diffusive interfaces by confining the moduli of $\theta$ and $\sigma$ to be less than unity. Using the heat-flux law obtained by combining (1), (3) and (4), we define a non-dimensional time

$$\tau = 3\cdot8\, ck_T\, T^{7/3}S^{-2}h^{-1}(\alpha/\beta)^2(g\alpha/k_T\nu)^{1/3}t \tag{6}$$

to obtain

$$\frac{d\theta}{d\tau} = (1-\theta)^{10/3}(1-\sigma)^{-2} - (1+\theta)^{10/3}(1+\sigma)^{-2}, \tag{7}$$

where we have neglected a term $\theta \dfrac{dh}{d\tau}$ with respect to $h \dfrac{d\theta}{d\tau}$. In physical terms, we neglect the effect the convection may have in moving the interface, and concentrate solely on the change in the layer properties resulting from the transfer through the interface. A rigorous argument supporting this approximation cannot be given. Intuition and all laboratory experiments performed so far, suggest that such an assumption is valid, and we shall invoke it throughout this investigation. The existence of equilibrium solutions, for which $\dfrac{d}{d\tau} \equiv 0$ will, of course, not depend on this approximation. A numerical simulation of high Rayleigh number thermal convection with a moving interface is presently being undertaken (D. O. Gough, E. A. Spiegel, J. Toomre; private communication), and it is anticipated that the techniques employed therein will be used to study two-component systems. In writing down (7) we have assumed that the equilibrium flux laws can be used to describe time-dependent situations. This is valid if the typical overturning time scale for the convecting eddies is much less than $\tau$, defined by (6).

Temporarily restricting attention to situations where both interfaces are in the constant regime, we find that

$$\frac{d\sigma}{d\tau} = \gamma \frac{d\theta}{d\tau}, \tag{8}$$

where $\gamma = 0 \cdot 15 \, \alpha T/\beta S$. From (7) and (8), we deduce that equilibrium configurations are specified by

$$\sigma = \frac{(1+\theta)^{5/3} - (1-\theta)^{5/3}}{(1+\theta)^{5/3} + (1-\theta)^{5/3}} \equiv g(\theta) \tag{9}$$

$$\sim \tfrac{5}{3}\theta \qquad\qquad (\theta \to 0) \tag{10a}$$

$$\sim \pm 1 \mp 2^{-2/3}(1\mp\theta)^{5/3} \qquad (\theta \to \pm 1). \tag{10b, c}$$

The function $g(\theta)$ is displayed in Fig. 4. The points $[\theta, g(\theta)]$ represent configurations for which there are equal fluxes through the two interfaces, but generally *not* equal temperature and salinity differences.

Are these equilibrium positions stable? Will a small perturbation in either temperature or salinity (or both) cause the ensuing interfacial flux to restore the system to its initial position, or drive it permanently away from an equilibrium position? We test for stability of the autonomous system (7) and (8) by the usual method of determining the solutions of the equations of motion linearised about the position of equilibrium (see SAATY and BRAM, 1964 or BELLMAN, 1953 for a more complete description). Expanding the right-hand-side of (7) about the equilibrium curve (9),
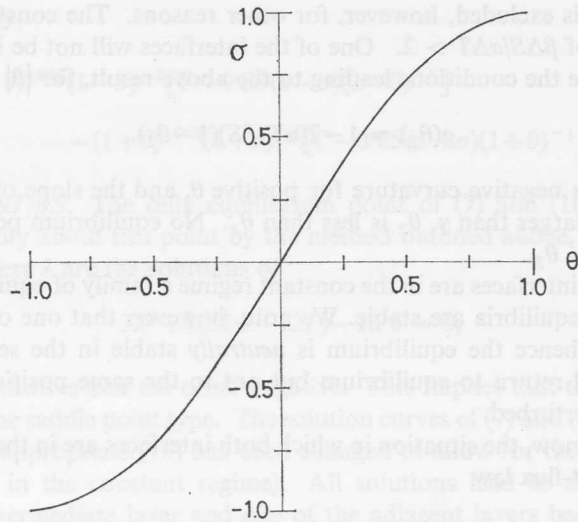
Fig. 4. The equilibrium positions $\sigma = g(\theta)$, as defined by (9).

linearising, denoting the coefficient of the $\theta$-variation by $\delta_\theta\{d\theta/d\tau\}$, and using similar notation for the $\sigma$-variation, we find that

$$\delta_\theta\left\{\frac{d\theta}{d\tau}\right\} = -\tfrac{5}{3}[(1+\theta)^{5/3}+(1-\theta)^{5/3}]^2(1-\theta^2)^{-1} = \gamma^{-1}\delta_\theta\left\{\frac{d\sigma}{d\tau}\right\}, \qquad (11a)$$

$$\delta_\sigma\left\{\frac{d\theta}{d\tau}\right\} = \tfrac{1}{4}[(1+\theta)^{5/3}+(1-\theta)^{5/3}]^4(1-\theta^2)^{-5/3} = \gamma^{-1}\delta_\sigma\left\{\frac{d\sigma}{d\tau}\right\}. \qquad (11b)$$

The perturbation solutions of (7) and (8) are of the form $e^{\lambda\tau}$, where $\lambda$ is an eigenvalue of

$$\mathbf{M} = \begin{bmatrix} \delta_\theta\left\{\dfrac{d\theta}{d\tau}\right\} & \delta_\sigma\left\{\dfrac{d\theta}{d\tau}\right\} \\[2ex] \delta_\theta\left\{\dfrac{d\sigma}{d\tau}\right\} & \delta_\sigma\left\{\dfrac{d\sigma}{d\tau}\right\} \end{bmatrix}. \qquad (12)$$

Substituting (11) into (12), we evaluate the eigenvalues as

$$0 \quad\text{and}\quad \gamma\delta_\sigma\left\{\frac{d\theta}{d\tau}\right\}+\delta_\theta\left\{\frac{d\theta}{d\tau}\right\}.$$

Hence $(\theta,\sigma)$ represents a position of stable equilibrium if and only if

$$\gamma \leqslant -\delta_\theta\left\{\frac{d\theta}{d\tau}\right\}\bigg/\delta_\sigma\left\{\frac{d\theta}{d\tau}\right\} \qquad (13a)$$

$$= \frac{20}{3}(1-\theta^2)^{2/3}[(1+\theta)^{5/3}+(1-\theta)^{5/3}]^{-2} \qquad (13b)$$

$$= g'(\theta). \qquad (13c)$$

From (13) we deduce that there could be instability only for $|\theta| > \theta_1$, where $g'(\theta_1) = \gamma$.

This range of $\theta$ is excluded, however, for other reasons. The constant regime exists only for values of $\beta \Delta S / \alpha \Delta T > 2$. One of the interfaces will not be in this range, and will hence violate the conditions leading to the above result, for $|\theta| > \theta_2$, where

$$g(\theta_2) = 1 - 2(\alpha T / \beta S)(1 - \theta_2). \tag{14}$$

Because $g(\theta)$ has negative curvature for positive $\theta$, and the slope of the straight line (14), $2\alpha T / \beta S$, is larger than $\gamma$, $\theta_2$ is less than $\theta_1$. No equilibrium position is, in fact, possible for $|\theta| > \theta_2$.

Thus, if both interfaces are in the constant regime a family of equilibrium positions exists and these equilibria are stable. We note, however, that one of the eigenvalues of **M** is 0, and hence the equilibrium is *neutrally* stable in the sense that a small perturbation will return to equilibrium but not to the same position from which it was originally perturbed.

Considering, now, the situation in which both interfaces are in the variable regime, we adopt the salt-flux law

$$\beta F_S / \alpha H = 1.85 - 0.85 \, \beta \Delta S / \alpha \Delta T \tag{15}$$
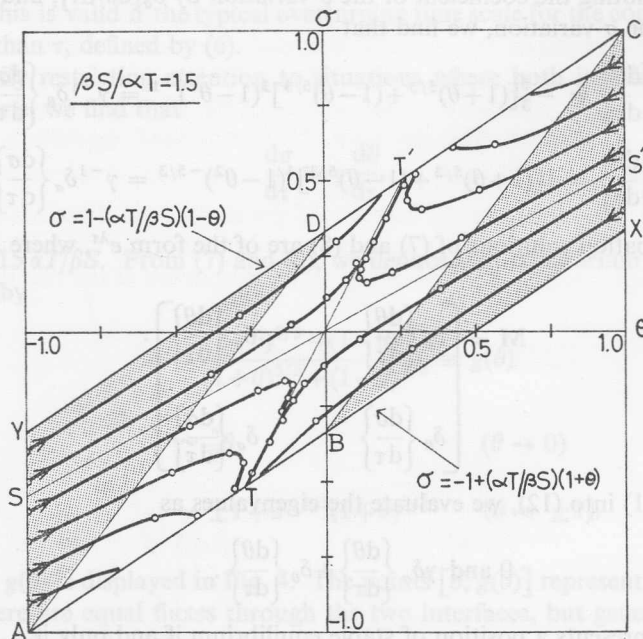


Fig. 5. The solution curves of (7) and (16) for $\beta S / \alpha T = 1.5$. The intervals between the open circles represent 0·2 units of the non-dimensional time defined by (6). The origin of time has been taken, for convenience, along AY and CX, with the direction of time indicated by the arrows. SS' and TT' are the separatrices of the system.

As the solution crosses the line YC the intermediate layer becomes heavier than the layer below it, the lower interface breaks down, and the adjacent layers merge. A similar mechanism involving the intermediate and uppermost layers occurs as the solution crosses the line AX.

Within the parallelogram ABCD both interfaces are in the variable regime. Within the dotted regions one of the interfaces is in the constant regime and the other is in the variable regime. The upper interface is in the constant regime within the left-hand dotted region and the lower interface is in the constant regime in the right-hand dotted region.

and replace (8) by

$$\frac{d\sigma}{d\tau} = (1-\theta)^{10/3}(1-\sigma)^{-2}[\Gamma - 0.85(1-\sigma)(1-\theta)^{-1}]$$
$$- (1+\theta)^{10/3}(1+\sigma)^{-2}[\Gamma - 0.85(1+\sigma)(1+\theta)^{-1}], \qquad (16)$$

where $\Gamma = 1.85\,\alpha T/\beta S$. The only equilibrium point of (7) and (16) is $\theta = \sigma = 0$. Testing for stability about this point by the method outlined above, we find that the stability parameters $\lambda$ are the solutions of

$$3\lambda^2 + \lambda(25.1 - 12\Gamma) - 13.6 = 0, \qquad (17)$$

one of which is positive and the other negative. This implies that the equilibrium is unstable and of the saddle point type. The solution curves of (7) and (16) are presented in Fig. 5 (where appropriate (16) has been changed to allow for the fact that one of the interfaces is in the constant regime). All solutions lead to overturning. The density of the intermediate layer and one of the adjacent layers becomes equal, the common interface breaks down, and two infinitely deep layers remain.
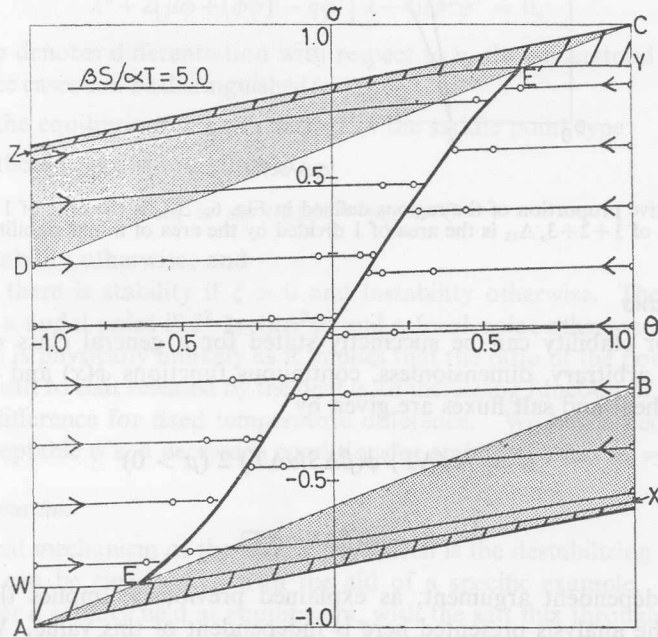


Fig. 6. The solution curves of (7) and (8) for $\beta S/\alpha T = 5.0$. The intervals between the open circles represent 0.2 units of the non-dimensional time defined by (6). The origin of time has been taken, for convenience, along AZ and CX, with the direction of time indicated by the arrows.

That part of the $\theta$, $\sigma$ plane for which the moduli of $\theta$ and $\sigma$ is less than 1 is divided into three regions: region 1, the curvilinear quadrilateral WXYZ, any point in which leads to the (stable) equilibrium curve EE'; region 2, the two hatched regions, which lead to overturning; and region 3, the remainder, for which the initial position violates static stability.

Within the parallelogram ABCD both interfaces are in the constant regime. Within the dotted regions one of the interfaces is in the variable regime and the other is in the constant regime. The lower interface is in the variable regime within the upper dotted region and the upper interface is in the variable regime within the lower dotted region.

The above analysis has concentrated upon *small* departures from equilibrium; large departures can also be examined. Consider, in particular, the system initiated at some point in the $\theta$, $\sigma$ plane with the modulus of both $\theta$ and $\sigma$ less than one. Does the subsequent motion, driven purely by the fluxes through the interfaces, tend to a position of equilibrium? If the *overall* stability ratio, $\beta S/\alpha T$, is less than 2, the question is answered in Fig. 5—in the negative. If $\beta S/\alpha T > 2$ the $\theta$, $\sigma$ plane is divided into three regions: region 1, from which a neutral equilibrium position is attained; region 2, which leads to overturning; and region 3, for which the initial position violates the static stability criterion. These regions are displayed in Fig. 6 for $\beta S/\alpha T = 5$, from which we see that for this value a major part of the plane is occupied by the first region. The proportion so occupied monotonically increases with $\beta S/\alpha T$ and is graphed in Fig. 7.
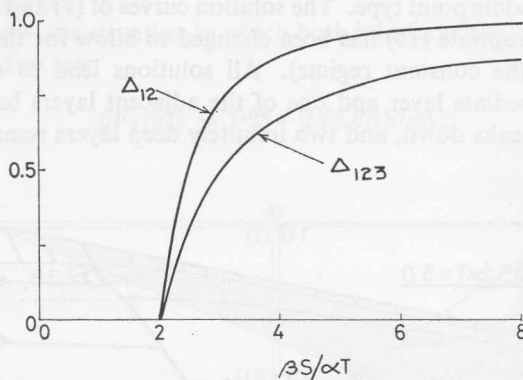


Fig. 7. The relative proportion of the regions defined in Fig. 6. $\Delta_{123}$ is the area of 1 divided by the total area of $1+2+3$, $\Delta_{12}$ is the area of 1 divided by the area of initial stability, $1+2$.

## General flux laws

Criteria for stability can be succinctly stated for a general class of flux laws. Consider two arbitrary, dimensionless, continuous functions $\phi(x)$ and $\psi(x)$ defined such that the heat and salt fluxes are given by

$$H = A \, (\Delta T)^{\mu} \phi(\beta \Delta S/\alpha \Delta T) \, 2 \, (\mu > 0) \tag{18a}$$

and

$$R = \psi(\beta \Delta S/\alpha \Delta T). \tag{18b}$$

The depth independent argument, as explained previously, implies that $\mu = 4/3$; nevertheless the analysis presented here is independent of this value. We constrain $\mu$ to be positive, however, since a non-positive value would imply that for fixed density ratio, the heat flux is a non-increasing function of the temperature difference, which would violate basic energetics. Defining a non-dimensional time $\tau'$ by

$$\tau' = A \, T^{\mu-1} h^{-1} t, \tag{19}$$

we apply (18) to the situation depicted in Fig. 3 to obtain (after dropping the prime associated with $\tau$)

$$\frac{d\theta}{d\tau} = (1-\theta)^\mu \phi[\eta(1-\sigma)(1-\theta)^{-1}] - (1+\theta)^\mu \phi[\eta(1+\sigma)(1+\theta)^{-1}] \quad (20a)$$

and

$$\eta \frac{d\sigma}{d\tau} = (1-\theta)^\mu \phi[\eta(1-\sigma)(1-\theta)^{-1}]\psi[\eta(1-\sigma)(1-\theta)^{-1}]$$
$$- (1+\theta)^\mu \phi[\eta(1+\sigma)(1+\theta)^{-1}]\psi[\eta(1+\sigma)(1+\theta)^{-1}], \quad (20b)$$

where

$$\eta = \beta S / \alpha T. \quad (21)$$

Except for special functions $\phi$ and $\psi$, $\psi$ equals a constant, for example, the only equilibrium point of (20) is $\theta = \sigma = 0$.* We analyse for stability about this point as before, to obtain the perturbation equations

$$\frac{d\theta}{d\tau} = 2(\eta\phi' - \mu\phi)\theta - 2\eta\phi'\sigma, \quad (22a)$$

$$\frac{d\sigma}{d\tau} = 2[(\phi\psi)' - \mu\eta^{-1}\phi\psi]\theta - 2(\phi\psi)'\sigma \quad (22b)$$

and the equation satisfied by the stability parameter $\lambda$:

$$\lambda^2 + 2[\mu\phi + (\phi\psi)' - \eta\phi']\lambda + 4\mu\phi^2\psi' = 0, \quad (23)$$

where a prime denotes differentiation with respect to $\eta$, the understood argument of $\phi$ and $\psi$. Three cases can be distinguished:

(i)  $\psi' < 0$, the equilibrium is unstable and of the saddle point type;

(ii) $\psi' = 0$, there is neutral stability if

$$\xi \equiv \mu\phi + (\phi\psi)' - \eta\phi' > 0 \quad (24)$$

and instability otherwise; and

(iii) $\psi' > 0$, there is stability if $\xi > 0$ and instability otherwise. The equilibrium point is a nodal point if $\xi^2 \geqslant 4\mu\phi^2\psi'$ and a focal point otherwise.

Condition (iii) is physically unlikely as it implies that the ratio of the potential energy gained by the salt to that released by the heat is an increasing function of the (stabilizing) salinity difference for fixed temperature difference. We hence deduce that for physically acceptable $\psi$'s, a necessary condition for stability is that $\psi' = 0$.

*Physical explanation*

The physical mechanism of the instability, which is the destabilizing action of the salinity field, can be clearly seen with the aid of a specific example. Referring to Fig. 8, consider a salinity field at equilibrium, with the salt flux through either interface marked by one of the 0's. We differentiate between case A for which $\psi' < 0$ and case B for which $\psi' = 0$ and perturb the salinity in the intermediate layer by a

---

*A sufficient condition for $\theta = \sigma = 0$ to be the only position of equilibrium is that $\psi$ be strictly monotonic. Because, equating the two terms on the right-hand-side of (20) and dividing the resulting equations, we deduce that the equilibrium curve is the solution of

$$(1-\theta)^\mu \phi[\eta(1-\sigma)(1-\theta)^{-1}] = (1+\theta)^\mu \phi[\eta(1+\sigma)(1+\theta)^{-1}]$$

and

$$\psi[\eta(1-\sigma)(1-\theta)^{-1}] = \psi[\eta(1+\sigma)(1+\theta)^{-1}].$$

If $\psi$ is strictly monotonic, the latter of these implies that $\theta = \sigma$, which, when substituted into the former implies that $\theta = \sigma = 0$.
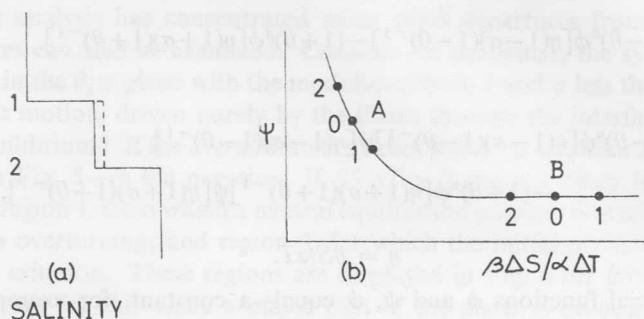
SALINITY

Fig. 8. (a) The salinity distribution at equilibrium plus a small, positive perturbation in the intermediate layer (dashed). (b) The function $\psi$ and the values taken by the interfaces. 0 represents the value taken by both interfaces in the equilibrium position. For case A $\psi' < 0$, while for case B $\psi' = 0$.

small, positive amount, as shown in Fig. 8a. Momentarily neglecting the ensuing temperature change, we mark the values of $\psi$ for the two interfaces on Fig. 8b. In case A, interface 2 transfers an increased amount of salt, while interface 1 transfers a decreased amount. Such changes are in the direction of *enlarging* the salinity perturbation. Further change involves considering both temperature and salinity fields, leading to the curves mapped out in Fig. 5 for particular functional forms of $\phi$ and $\psi$. In case B, there is no initial change in the salt flux, so the change in the heat flux and the resulting temperature change of the intermediate layer must be considered. From this we find that the temperature field counteracts the salinity variation, and the net result is given by the quantity $\xi$; if $\xi < 0$ the destabilizing effect of the salt exceeds the stabilizing effect of the temperature, while if $\xi > 0$ the reverse is true.

This describes the mechanism of instability for any initial perturbation. If we restrict both $\phi'$ and $(\phi\psi)'$ to be negative—for fixed temperature differences both the heat and salt flux are decreasing functions of the salinity difference—the coefficients of the $\theta$ variations in (22) are negative, those of the $\sigma$ variations positive. The salinity is hence the destabilizing agent, opposing the stabilizing effect of the temperature. If $\psi' < 0$, the effect of the salt is always dominant. For $\psi' \geqslant 0$ the net effect of the two components is expressed by the quantity $\xi$.

## THE STABILITY OF MANY INTERMEDIATE LAYERS

The concepts introduced in the previous section can be simply extended to encompass a *series* of intermediate layers. Consider $N$ such layers of thickness $hd_i$, temperature $T\theta_i$ and salinity $S\sigma_i$ ($i = 1, 2, \ldots, N$, the layer numbers increasing with depth). The series of layers is bounded above by an infinite layer of temperature and salinity 0 and below by one of temperature $T$, salinity $S$. We still restrict attention to diffusive interfaces by specifying that $\theta_{i+1} > \theta_i$, $\sigma_{i+1} > \sigma_i$ 2 ($\theta_0 = \sigma_0 = 0$, $\theta_{N+1} = \sigma_{N+1} = 1$).

Using the flux formulae (18) and the non-dimensional time defined by (19), we deduce that

$$d_i \frac{d\theta_i}{d\tau} = (\theta_{i+1} - \theta_i)^\mu \phi_{i+1} - (\theta_i - \theta_{i-1})^\mu \phi_i \tag{25a}$$

and

$$\eta d_i \frac{d\sigma_i}{d\tau} = (\theta_{i+1} - \theta_i)^\mu \phi_{i+1} \psi_{i+1} - (\theta_i - \theta_{i-1})^\mu \phi_i \psi_i, \tag{25b}$$

where the subscript $i$ implies evaluation for the function at $\eta(\sigma_i - \sigma_{i-1})(\theta_i - \theta_{i-1})^{-1}$. From the similarity of (20) and (25) one would anticipate that the conclusions for $N$ intermediate layers mirror those for one, and indeed this is so.

The positions of equilibrium are given by

$$(\theta_{i+1} - \theta_i)^\mu \phi_{i+1} = (\theta_i - \theta_{i-1})^\mu \phi_i \quad \text{and} \quad \psi_{i+1} = \psi_i, \tag{26a, b}$$

which for strictly monotonic $\psi$ have only the solution

$$\theta_i = \sigma_i = i/(N+1). \tag{27}$$

For constant $\psi$, the solution is of the form $\sigma_i = \sigma_i(\theta_1, \theta_2, \ldots, \theta_N)$. For example, for $\mu = 4/3$, $\phi(x) \propto x^{-2}$ and $\psi(x)$ constant, the solution of (26) is

$$\sigma_i = \sum_{r+1}^{i} (\theta_r - \theta_{r-1})^{5/3} \Bigg/ \sum_{r+1}^{N+1} (\theta_r - \theta_{r-1})^{5/3}. \tag{28}$$

We immediately deduce from (25) that the equilibrium positions are independent of the individual thickness; in an equilibrium position each layer may be of any thickness. The nature of the stability of these equilibrium positions is exactly as outlined above and, as before, $\psi' < 0$ implies instability, and $\psi' \geqslant 0$ implies (in)stability if $\xi(\leqslant) > 0$, independently of the layer thicknesses. A rigorous proof of this statement is presented in the Appendix.

This extension, combined with the physical explanation of the instability, implies that it is a local effect. A number of layers belonging to a larger series may merge, independently of the conditions many layers away.

*Fingering interfaces*

We have so far concentrated attention on diffusive interfaces solely because there are a large number of controlled, laboratory experiments for this case. The adaption of the above presentation to a fingering interface is immediate. We need only invert Fig. 3 to consider hot, salty fluid above cold, fresh fluid and everywhere interchange 'heat' quantities, $H$, $\alpha$, $T$, $\ldots$, and 'salt' quantities, $F_S$, $\beta$, $S$, $\ldots$; in particular $\eta$ becomes $\alpha T/\beta S$. The general stability conditions are then exactly as above, the $\phi$ and $\psi$ being considered as functions of this altered $\eta$, and instability is due to the destabilizing action of the temperature field. A sufficient number of experiments have not yet been undertaken to completely specify the functional form of $\phi$ and $\psi$. It is known, however, (TURNER, 1967) that $\psi$ is a constant, 0·56, for $2 < \alpha\Delta T/\beta\Delta S < 10$. Further experiments are presently being planned at the University of Cambridge.

### CONCLUSIONS AND OCEANOGRAPHIC IMPLICATIONS

We have shown that the stability characteristics of a series of double-diffusive layers can be expressed in terms of the functions $\phi(\eta)$, $\psi(\eta)$, and $\xi(\eta) = \mu\phi(\eta) + [\phi(\eta)\psi(\eta)]' - \eta\phi'(\eta)$, where $\eta$ is the ratio of the *overall* density change resulting from the stabilizing component to that resulting from the destabilizing component.

If $\psi$ is constant, a continuum of equilibrium positions exists. In these positions for a given component there is the same flux through each interface, but not necessarily the same increment across each interface. The layer depths need not be equal. The series of layers is stable for $\xi > 0$ and unstable otherwise.

If $\psi'$ is negative, equilibrium occurs only if there is the same increment of a given component across each interface. The equilibrium is always unstable. The density of adjacent layers becomes equal, the common interface breaks down, and the two layers merge. This continues throughout the whole series of layers.

If $\psi'$ is positive, a condition not yet observed in any laboratory experiment and physically unlikely, the position of equilibrium is as for negative $\psi'$, and the layers are (un)stable if $\xi > (\leqslant 0)$.

Oceanographic observations of double-diffusive layering to which the instability theory would apply are now quite plentiful. A clear set of some dozen diffusive interface has been observed by HOARE (1966, 1968) and SHIRTCLIFFE and CALHAEM (1968) in Lake Vanda, an Antarctic lake of approximately $6 \, \text{km} \times 1 \, \text{km} \times 65 \, \text{m}$. The temperature increases, in steps, from $0° \, \text{C}$ below the 3-metre thick ice cover to $25°\text{C}$ at the bottom. There is a corresponding salinity increase from 0 to $10\%$ by weight. From the quoted measurements, we calculate $\beta S/\alpha T$ to be 10. We would thus expect the layers to be stable, and indeed they have been consistently observed during the past decade. An exciting series of measurements has also been undertaken in the hot brine layers recently discovered at the bottom of the Red Sea. Much of the work is reported in DEGENS and ROSS (1969). Using data presented therein, we calculate the various values of $\beta S/\alpha T$ all to exceed 10.

Layers with hot, salty water overlying cold, fresh water have been observed by COOPER and STOMMEL (1968) near Bermuda, by AMOS (private communication) off Barbados and by TAIT and HOWE (1968) beneath the Mediterranean outflow. The value of $\alpha T/\beta S$ extracted from Tait and Howe's measurements is $1\cdot15$. The implication is that either $\psi(\eta)$ is constant for $\eta > 1\cdot15$, or the time taken for the instability was sufficiently large that the series of layers did not appreciably vary during the time the measurements were taken. The calculation of a time for the instability requires more specific knowledge of $\phi$ and $\psi$ for fingering interfaces than presently available, and further, the answer depends markedly on the initial perturbation, as can be seen from Fig. 5. A time *scale* can be obtained from (6), after replacing $3\cdot8$ by $50-$ Turner's experiments indicate that $\phi(1) \approx 50$ for a salt-fingering interface — setting $\eta = 1$ therein and interchanging all heat and salt values as indicated above. Using the values $c = 0\cdot085$, $k_S = 10^{-5}$, $h = 2\cdot2 \times 10^3$, $\beta = 0\cdot66$, $\alpha = 9 \times 10^{-5}$, $g = 980$, $v = 1\cdot3 \times 10^{-2}$ (all in c.g.s. units), $T = 1\cdot1°\text{C}$ and $S = 0\cdot18\%_0$, we find that

$$\tau = 1\cdot4 \times 10^{-6}t.$$

Thus, the instability, if it exists, has a time *scale* of approximately 5 days. Tait and Howe's measurements extended over a considerably shorter time than this.

### APPENDIX

We prove in this Appendix that the conditions governing stability of (25) about the equilibrium point (27) are exactly those governing the stability of (20) about the equilibrium $\theta = \sigma = 0$.

Appropriately differentiating (25) and substituting (27) into the result, we obtain as the determinental equation for the stability parameter $\lambda$

$$|\mathbf{B} - \mathbf{H}\lambda| = 0, \tag{A1}$$

where

$$\mathbf{B} = \begin{bmatrix} \mathbf{C}_1 & \mathbf{C}_3 \\ \mathbf{C}_2 & \mathbf{C}_4 \end{bmatrix} \tag{A2}$$

and

$$\mathbf{H} = \begin{bmatrix} \mathbf{D} & \mathbf{0} \\ \mathbf{0} & \mathbf{D} \end{bmatrix} \tag{A3}$$

are $2N \times 2N$ matrices given in terms of the $N \times N$ matrices

$$\mathbf{D} = \begin{bmatrix} d_1 & 0 & 0 & 0 & . \\ 0 & d_2 & 0 & 0 & . \\ 0 & 0 & d_3 & 0 & . \\ 0 & 0 & 0 & d_4 & . \\ . & . & . & . & . \end{bmatrix} \tag{A4}$$

and

$$\mathbf{C}_i = c_i \begin{bmatrix} 2 & -1 & 0 & 0 & . \\ -1 & 2 & -1 & 0 & . \\ 0 & -1 & 2 & -1 & . \\ 0 & 0 & -1 & 2 & . \\ . & . & . & . & . \end{bmatrix} \equiv c_i \mathbf{E} \quad (i = 1, 4), \tag{A5}$$

with

$$c_1 = \eta\phi' - \mu\phi, \tag{A6a}$$

$$c_2 = -\eta\phi', \tag{A6b}$$

$$c_3 = (\phi\psi)' - \mu\eta^{-1}\phi\psi, \tag{A6c}$$

and

$$c_4 = -(\phi\psi)', \tag{A6d}$$

each function being evaluated at $\eta$.

Defining the Kronecker product, $\mathbf{A} \otimes \mathbf{B}$, of two matrices $\mathbf{A}$ and $\mathbf{B}$ by (MARCUS and MINC, 1964, p. 8)

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & a_{13}\mathbf{B} & . \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} & a_{23}\mathbf{B} & . \\ a_{31}\mathbf{B} & a_{32}\mathbf{B} & a_{33}\mathbf{B} & . \\ . & . & . & . \end{bmatrix}, \tag{A7}$$

we write (A1) in the form

$$\mathbf{B}x = \mathbf{L} \otimes \mathbf{E}x = \lambda\mathbf{H}x, \tag{A8}$$

where

$$\mathbf{L} = \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix}. \tag{A9}$$

A direct consequence of the definition of the Kronecker product is that if the vectors $\mathbf{v}$ and $\mathbf{w}$ and the corresponding numbers $\alpha$ and $\beta$ satisfy

$$\mathbf{Lv} = \alpha\mathbf{v} \quad \text{and} \quad \mathbf{Ew} = \beta\mathbf{Hw}_i \qquad \text{(A10a, b)}$$

then

$$(\mathbf{L} \otimes \mathbf{E})(\mathbf{v} \otimes \mathbf{w}) = \mathbf{Lv} \otimes \mathbf{Ew} \qquad \text{(A11a)}$$

$$= \alpha\mathbf{v} \otimes \beta\mathbf{Hw} \qquad \text{(A11b)}$$

$$= \alpha\beta\mathbf{H}(\mathbf{v} \otimes \mathbf{w}). \qquad \text{(A11c)}$$

The $\lambda$'s satisfying (A8) are hence the product of the $\alpha$'s and $\beta$'s satisfying (A10). The latter are (real and) positive, since from (A10b)

$$\beta = \frac{\mathbf{w}^T\mathbf{Ew}}{\mathbf{w}^T\mathbf{Hw}}, \qquad \text{(A12)}$$

and both numerator and denominator are positive quadratic forms ($\mathbf{w}^T\mathbf{Ew}$ is positive definite because all the eigenvalues of $\mathbf{E}$, $4\cos^2\left[\frac{1}{2}s\pi/(N+1)\right]$ $(s = 1, 2, \ldots, N)$ are positive). Thus the sign of $\lambda$ is equal to that of $\alpha$, the eigenvalue of $\mathbf{L}$, which satisfies (23).

The stability conditions for $N$ intermediate layers of arbitrary thickness are hence exactly the same as those for one.

REFERENCES

BELLMAN R. (1953) *Stability theory of differential equations.* McGraw-Hill, 166 pp.
COOPER J. W. and H. STOMMEL (1968) Regularly spaced steps in the main thermocline near Bermuda. *J. geophys. Res.*, 73, 5849–5854.
DEGENS E. T. and D. A. ROSS, editors (1969) *Hot brines and Recent heavy metal deposits in the Red Sea.* Springer, 600 pp.
HERRING J. R. (1963) Investigation of problems in thermal convection. *J. Met.*, 20, 325–338.
HERRING J. R. (1964) Investigation of problems in thermal convection: rigid boundaries. *J. Met.*, 21, 277–290.
HOARE R. A. (1966) Problems of heat transfer in Lake Vanda, a density stratified Antarctic lake. *Nature, Lond.*, 210, 787–789.
HOARE R. A. (1968) Thermohaline convection in Lake Vanda, Antarctica. *J. geophys. Res.*, 73, 607–612.
HOWARD L. N. (1966) Convection at high Rayleigh number, pp. 1190–1115. In: *Proceedings 11th international congress of applied mechanics*, Munich, 1964, H. GÖRTLER, editor, Springer, 1189 pp.
MARCUS M. and H. MINC (1964) *A survey of matrix theory and matrix inequalities.* Allyn and Bacon, 180 pp.
ROSSBY H. T. (1966) *An experimental study of Bénard convection with and without rotation.* Scientific Report HRF/SR27, Hydrodynamics of Rotating Fluids Project, Department of Geology and Geophysics, M.I.T.
ROSSBY H. T. (1969) A study of Bénard convection with and without rotation. *J. Fluid Mech.*, 36, 309–335.
SAATY T. L. and J. BRAM (1964) *Nonlinear mathematics.* McGraw-Hill, 381 pp.
SHIRTCLIFFE T. G. L. (1967) Thermosolutal convection: observation of an overstable mode. *Nature, Lond.*, 213, 489–490.

SHIRTCLIFFE T. G. L. and I. M. CALHAEM (1968) Measurements of temperature and electrical conductivity in Lake Vanda, Victoria Land, Antarctica. *N.Z. Jl. Geol. Geophys.*, **11**, 976–981.

SPIEGEL E. A. (1967) The solar atmospheric convection zone, pp. 347–366 In: *Fifth symposium on cosmological gas dynamics. Aerodynamic phenomena in stellar atmospheres*, Nice, 1965, R. N. THOMAS, editor, Academic Press, 483 pp.

STERN M. E. (1969) Collective instability of salt fingers. *J. Fluid Mech.*, **35**, 209–218.

STERN M. E. and J. S. TURNER (1969) Salt fingers and convecting layers.    *Deep-Sea Res.* **16**, 497–511.

TAIT R. I. and M. R. HOWE (1968) Some observations of thermo-haline stratification in the deep ocean. *Deep-Sea Res.*, **15**, 275–280.

TURNER J. S. (1965) The coupled turbulent transports of salt and heat across a sharp density interface. *Int. J. Heat Mass Transfer*, **8**, 759–767.

TURNER J. S. (1967) Salt fingers across a density interface. *Deep-Sea Res.*, **14**, 599–611.

VERONIS G. (1965) On finite amplitude instability in thermohaline convection. *J. mar. Res.*, **23**, 1–17.