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# On the thermal evolution of the Earth's core

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**Abstract.** The Earth's magnetic field is sustained by dynamo action in the fluid outer core. The energy sources available to the geodynamo are well established, but their relative importance remains uncertain. We focus on the issue of thermal versus compositional convection, which is inextricably coupled to the evolution of the core as the Earth cools. To investigate the effect of the various physical processes on this evolution, we develop models based on conservation of energy and the assumption that the core is well mixed by vigorous convection. We depart from previous numerical studies by developing an analytical model. The simple algebraic form of the solution affords insight into both the evolution of the core and the energy budget of the geodynamo. We also present a numerical model to compare with the quantitative predictions of our analytical model and find that the differences between the two are negligible. An important conclusion of this study is that thermal convection can contribute significantly to the geodynamo. In fact, a modest heat flux in excess of that conducted down the adiabatic gradient is sufficient to power the geodynamo, even in the absence of compositional convection and latent-heat release. The relative contributions of thermal and compositional convection to the dynamo are largely determined by the magnitude of the heat flux from the core and the inner-core radius. For a plausible current-day heat flux of $Q = 3.0 \times 10^{12}$ W and the current inner-core radius, we find that compositional convection is responsible for approximately two thirds of the ohmic dissipation in the core and thermal convection for the remaining one third. The proportion of ohmic dissipation produced by thermal convection increases to 45% with an increase in $Q$ to $6.0 \times 10^{12}$ W. In the early Earth, when the inner core was smaller and the heat flux probably greater than the present values, thermal convection would have been the dominant energy source for the dynamo. We also calculate the history of inner-core growth as a function of the heat flux. For example, the inner core would have grown to its present size in $2.8 \times 10^9$ years if the average heat flux was $Q = 4.0 \times 10^{12}$ W. The model does not require the heat flux to be constant.

## 1. Introduction

Paleomagnetic measurements show that the Earth has possessed a magnetic field for at least 3 billion years [*McElhinny and Senanayake*, 1980]. The persistence of the field provides compelling evidence for a dynamo process because the field would otherwise have dissipated through ohmic losses in less than $10^5$ years [e.g., *Stevenson*, 1974]. Although the precise operation of the dynamo is poorly understood, it is generally believed to be driven by vigorous convection in the electrically conducting, fluid outer core. A supply of energy is needed to power the dynamo, and the most important sources are usually associated with the cooling and gradual solidification of the core [*Gubbins*, 1977; *Loper*, 1978a, 1991; *Gubbins et al.*, 1979]. Some of the energy available to the dynamo arises through latent-heat release on solidification [*Verhoogen*, 1961] and through the associated generation of compositional buoyancy [*Braginsky*, 1963] by the exclusion of light elements from the iron-rich solid inner core. Other possible sources of energy include the thermal energy associated with cooling, the gravitational energy released by thermal contraction, radioactive sources, and precession [*Malkus*, 1963; *Rochester et al.*, 1975].

The relative importance of the various energy sources in the dynamo problem remains an important question [e.g., *Verhoogen*, 1980]. Calculations of the thermal history of the core [e.g., *Stevenson et al.*, 1983; *Mollett*, 1984; *Stacey and Loper*, 1984] typically indicate that the gravitational energy released by compositional con-

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vection is the most important source of energy for the geodynamo, and it is sometimes thought to be essential. Thermal convection has been thought to play a small role because of the low efficiency of a Carnot engine at core temperatures, but this view does not fully take into account the effects of latent heat and the relative magnitudes of the gravitational energy released by the thermal and compositional buoyancy fluxes.

Gravitational energy may be released in a variety of ways and it is important to distinguish between them. *Gubbins et al.* [1979] showed that the gravitational energy released by slow cooling and contraction of the core through a sequence of hydrostatic equilibria is wholly converted to internal energy of compression (sometimes referred to as deformational energy [*Häge and Müller*, 1979]). Thus the gravitational energy released nonconvectively by thermal conduction along the radial temperature gradient is not available to the geodynamo. On the other hand, we show here that convective cooling by the downward mixing of fluid from a cold thermal boundary layer at the core-mantle boundary (CMB) produces a significant contribution to the geodynamo due to the small departures from hydrostatic equilibrium inherent in convection. Similarly, a purely barodiffusive segregation of the core into light and heavy components would not contribute to the geodynamo (even if it were rapid enough), whereas upward convective mixing of light fluid from the compositional boundary layer at the inner-core boundary (ICB) does. Hence, in order to assess the energy supply for the geodynamo, we must identify that part of the gravitational energy which is released by the thermal and compositional buoyancy fluxes associated respectively with thermal and compositional convection.

The thermal evolution of the core involves a variety of coupled physical processes which effect the response of the core to the heat flux across the CMB. As heat is extracted from the core, the temperature at the ICB decreases, causing solidification and latent-heat release. The cooling and contraction of the core releases gravitational energy, part of which may be converted to heat by thermal convection and ohmic dissipation in the core. Further gravitational energy is released by the exclusion of light elements from the solid inner core, which generates compositional buoyancy and also drives convection and produces heat through ohmic dissipation. Light elements and heat are also transported by diffusion, but without generating ohmic dissipation. The fractionation of light elements into the outer core affects the liquidus temperature, which is a function of both pressure and composition. In addition, the geometry evolves as the radius of the solid inner core gradually increases. The interaction of these various physical processes in the global heat balance can be quantified by determining the rate of convective gravitational energy release and the evolution of the temperature, pressure and composition.

There are two main purposes of this paper. The first is to calculate the thermal evolution of the core, and the second is to reassess the energy sources available to the geodynamo. The global energy budget of the core is described in section 2. Thermal and compositional convection are argued to play entirely analogous roles in the dynamo, but neither appear explicitly in the global energy budget of the core since the ohmic dissipation associated with convection occurs primarily within the core itself. Instead, the global energy budget reveals only the net cooling and compositional segregation of the core by the combined effects of convection and diffusion. An analytical model that predicts the thermal evolution of the vigorously convecting, outer core and the concomitant growth of the inner core is developed in section 3. The analytical formulation gives results in a simple algebraic form, as exemplified by equations (33)–(38) for the growth of the inner core. The algebraic form of the results allows the consequences of changing the model input parameters to be readily assessed, which is an advantage over previous numerical calculations, and also affords insights into the relative importance of the processes that affect the geodynamo energy budget and inner-core growth. The accuracy of a number of simplifying approximations made in the analytical model is tested in section 4 by comparing the results with a more general numerical calculation. The errors introduced by the approximations are found to be negligible given the present uncertainties in the thermodynamic properties of the Earth.

Reconsideration of the gravitational energy released through thermal convection and the associated departures from a perfectly hydrostatic state eliminates objections previously raised against a thermally-driven dynamo [*Gubbins*, 1977; *Gubbins et al.*, 1979]. A modest heat flux in excess of that conducted down the adiabatic gradient is shown in section 5 to be sufficient to sustain a geodynamo, even in the absence of compositional convection and latent-heat release. When compositional convection is also taking place, as at present, then thermal convection can still make a significant contribution to the total energy of the dynamo. The conclusions are discussed in section 6 and further implications of our model for the early Earth and Venus are noted.

## 2. Global Energy Balance

### 2.1. Basic Equations

Our analysis of the thermal evolution of the core is based on conservation of energy. The relevant equations are derived from integration over the core of the equations that describe local changes of momentum and internal energy, together with Maxwell's equations for electromagnetic induction [e.g., *Backus*, 1975; *Hewitt et al.*, 1975; *Verhoogen*, 1980]. Denote the volume of the core by $V$, which is enclosed by the surface $S$ with outward normal $\mathbf{dS}$, and assume that there is no mass transport across the CMB so $V$ is a material volume. The fluid within the core has a velocity $\mathbf{v}$ and is slowly cooling due to the heat flux $\mathbf{q}$ into the base of the mantle. Fluid motion sustains an electric current density $\mathbf{J}$ through dynamo action, which produces the magnetic field $\mathbf{B}$ and a Lorentz force $\mathbf{F} = \mathbf{J} \times \mathbf{B}$. Since the core contracts as it cools, $V$ changes with time and the normal component of $\mathbf{v}$ on $S$ is nonzero.

For simplicity, we assume that the effect of viscosity in the core is negligible, so that mechanical energy is dissipated entirely by ohmic losses. In this approximation, the rate of change of mechanical energy, as obtained from the local momentum equation, can be written in the form [*Chandrasekhar*, 1981]

$$\frac{1}{2}\frac{d}{dt}\int_V \rho v^2\, dV = \int_V \mathbf{v}\cdot(\mathbf{F} - \rho\nabla\psi - \nabla P)\, dV\,, \quad (1)$$

where $\rho$ is the fluid density, $P$ is the pressure and $\psi$ is the gravitational potential.

From Maxwell's equations with the usual magnetohydrodynamic approximation [*Elsasser*, 1946; *Bullard and Gellman*, 1954], the rate of change of magnetic energy is given by

$$\frac{1}{2}\frac{d}{dt}\int_{V_\infty} \frac{B^2}{\mu_0}\, dV = -\int_V \mathbf{v}\cdot\mathbf{F}\, dV - \int_V \frac{J^2}{\sigma}\, dV\,, \quad (2)$$

where $\sigma$ is the electrical conductivity and $\mu_0 = 4\pi \times 10^{-7}$ H m$^{-1}$ is the permeability of free space. In (2) the change in magnetic energy is evaluated over all space $V_\infty$, whereas the currents, and hence the Lorentz forces, are restricted to the volume $V$ of the electrically conducting core under the assumption that the electrical conductivity of the mantle is negligible.

The internal energy budget of the core is given by [*Tritton*, 1988, p. 191]

$$\frac{d}{dt}\int_V \rho u\, dV = -\int_S \mathbf{q}\cdot d\mathbf{S} + \int_V \rho h\, dV + \int_V \frac{J^2}{\sigma}\, dV - \int_V P(\nabla\cdot\mathbf{v})\, dV\,, \quad (3)$$

where $u$ is the specific internal energy and $h$ is the rate of radioactive heating per unit mass. We will not include $h$ in the subsequent discussion since it can be eliminated from (3) by subtracting its contribution from the heat flux across the CMB. Equation (3) may be manipulated into a more convenient form by using the thermodynamic relationship and equations for conservation of mass and light element,

$$du = T ds + \frac{P}{\rho^2} d\rho + \mu dC \quad (4)$$

$$\frac{D\rho}{Dt} = -\rho(\nabla\cdot\mathbf{v}) \quad (5)$$

$$\rho\frac{DC}{Dt} = -\nabla\cdot\mathbf{i}\,, \quad (6)$$

where $T$ is the absolute temperature, $s$ is the specific entropy, $C$ is the concentration of the light element in the core, $\mu$ is the chemical potential of the mixture, $D/Dt$ is the material derivative [e.g., *Tritton*, 1988, p. 54] and $\mathbf{i}$ is the diffusional flux of the light element. The result of these manipulations is

$$\int_V \rho T\frac{Ds}{Dt} dV = -\int_S \mathbf{q}\cdot d\mathbf{S} + \int_V \frac{J^2}{\sigma}\, dV + \int_V \mu\nabla\cdot\mathbf{i}\, dV\,, \quad (7)$$

which represents the heat budget of the core. Equation (7) states that changes in the heat content of the core are due to heat flow from the core, to generation of ohmic heat in the interior and to any diffusive segregation of light element.

When describing the thermal evolution of the core, (1) and (2) can be simplified further since the changes in the magnetic and kinetic energies are negligible in comparison with the leading-order changes in the gravitational and internal energies [*Gubbins*, 1977]. On dropping these small terms and eliminating the work done against Lorentz forces between (1) and (2), the equation for the rate of change of mechanical energy reduces to

$$\int_V \mathbf{v}\cdot(\rho\nabla\psi + \nabla P)\, dV = -\int_V \frac{J^2}{\sigma}\, dV \equiv \Phi\,, \quad (8)$$

where $\Phi$ is the total ohmic dissipation. Equation (8) states that the ohmic dissipation associated with dynamo action is the difference between the change in gravitational energy and the work done against pressure gradients.

## 2.2. Does Thermal Convection Contribute to the Dynamo?

Gravitational energy is released by net thermal contraction and compositional fractionation due to the slow cooling of the core as described by (7), together with a consequent increase in the compression of the core as the radial mass distribution changes. If the pressure in the core were perfectly hydrostatic, that is $\nabla P = -\rho\nabla\psi$, then from (8) the ohmic heat generation would be zero. Thus the gravitational energy released in a perfectly hydrostatic core would be entirely converted to internal energy of compression with a small amount of compressional (adiabatic) heating, and none would be available to power the geodynamo. Hence the ohmic heating is maintained solely by departures of the core from a hydrostatic state.

The nonhydrostatic pressure gradients are associated with convection driven by cooling from above and light-element release from below. Convection converts some of the gravitational potential energy through motion and ohmic dissipation to heat. Thus the energy source for the ohmic heating in (8) is precisely that part of the gravitational energy which is converted to motion by thermal and compositional convection. Additional gravitational energy may be released by diffusion of heat and light elements along the adiabat and by compression of the core as the radial mass distribution changes, but such processes do not perturb the nearly hydrostatic balance in the core and, consequently, do not contribute to the ohmic heating. We must therefore distinguish between that part of the gravitational energy which is converted to heat by convection and the consequent ohmic dissipation, and that part which is converted directly to internal energy of compression by diffusion and adjustment of the radial mass distribution.

The need to eliminate compressional energy from the energy budget of the dynamo was noted by *Gubbins et al.* [1979], who sought estimates for the proportion of the gravitational energy release that is converted to ohmic dissipation by convection. They proposed that the gravitational energy released by compositional rearrangement is fully converted to ohmic dissipation, whereas none of the gravitational energy released by thermal convection is converted to heat. This result would be surprising because the expression for the gravitational energy release in (8) does not distinguish between the energy released by thermal convection and by compositional rearrangement. We attribute the conclusions of *Gubbins et al.* [1979] to use of a hydrostatic assumption in their analysis of thermal convection, which ensured that all of the associated gravitational energy release appeared as internal energy of compression. However, perfect hydrostatic equilibrium is inconsistent with the presence of convective flow. Provided the heat flux from the core exceeds that conducted along the adiabatic gradient, cold, dense fluid from an unstable thermal boundary layer at the CMB descends into the underlying fluid, thus releasing gravitational energy in a manner completely analogous to that due to compositional convection driven by light fluid rising from the compositional boundary layer at the ICB. We show later that, even if the core heat flux is subadiabatic at the CMB, thermal convection can still contribute to the dynamo due to the thermal buoyancy flux associated with latent-heat release deeper in the outer core.

Since there is no mechanical difference between thermal and compositional convection, as can be seen in (1) or (8), their relative importance is determined by the magnitude of the associated thermal and compositional buoyancy fluxes, which must be calculated from the rates of cooling and solidification of the core. We show in section 5 that, for plausible convective heat fluxes in the core, thermal convection makes a significant contribution to the dynamo, particularly early in Earth history.

### 2.3. Does Thermal Convection Contribute to the Heat Budget?

While thermal and compositional convection play equivalent mechanical roles in powering a dynamo, some differences emerge in the form of their contribution to the total heat budget (7) through the generation of ohmic heat. The rate of change $\dot{\Sigma}$ of the total energy within the core is obtained by adding (1)–(3) to yield

$$\dot{\Sigma} \equiv \frac{d}{dt}\int_V \rho u \, dV + \int_V \rho \mathbf{v} \cdot \nabla \psi \, dV = -\int_S \mathbf{q} \cdot \mathbf{dS} - \int_S P\mathbf{v} \cdot \mathbf{dS}, \qquad (9)$$

where changes in magnetic, kinetic and nuclear energies are omitted from $\dot{\Sigma}$ under the approximations described. The work done against Lorentz forces, the ohmic dissipation and the internal pressure work associated with $P\nabla \cdot \mathbf{v}$ do not appear because these merely correspond to conversion of energy within the core from one form to another with the total energy conserved. Indeed, the global energy equation (9) does not reveal any internal details of the convective and diffusive processes within the core because the changes in both gravitational energy and internal energy are total differentials in the usual sense that differences in these quantities between any two states do not depend on the path taken between them.

The thermodynamic and gravitational state of the core is described by the temperature, pressure, composition and density fields, which, to within the small convective fluctuations, can be determined from the conditions of a well-mixed, adiabatic and hydrostatic core, together with a suitable equation of state and a melting curve for the core liquid. (A well-mixed, adiabatic state is also isentropic [*Gubbins et al.*, 1979].) As the core cools on a timescale much longer than that of convective fluctuations, the mean conditions in the core evolve through a succession of such states parameterized, say, by the inner-core radius. Thus the change of $\Sigma$ between two mean states is independent of whether that change took place by thermal and compositional convection or by static cooling and barodiffusion, provided that the mean conditions in the core remain approximately well-mixed, adiabatic and hydrostatic. Moreover, since the pressure work on the CMB can be evaluated from the differences in the two states, the total amount of heat removed from the core during the transition between the states can be determined from (9) and is also independent of the details of the internal processes. How then is the dissipation associated with the thermal and compositional convection manifested in the heat balance (7)? In order to address this question, we consider the effects of thermal and compositional convection separately in the following two paragraphs though, of course, they occur concurrently. (This separate consideration is done in the usual spirit of thermodynamic arguments that decompose a given change in some system into a number of simpler path segments in state space.)

If the core were to cool diffusively through a series of perfectly hydrostatic states, then (8) shows that the gravitational energy released by thermal contraction is taken up as internal energy of compression and $\Phi = 0$. On the other hand, if the core were to cool by thermal convection between the same two states, then the changes in gravitational and internal energy would be the same as before and so the heat loss across the CMB must also be the same. We conclude that the contribution of thermal convection alone to the ohmic dissipation on the right-hand side of (7) must be exactly balanced by the advective term $\rho T \mathbf{v} \cdot \nabla s$ on the left-hand side of (7) which is due to the convection. This term corresponds to the heat absorbed due to the convection and mixing of entropy anomalies through the core which maintains an adiabatic state. If the vigour of thermal convection and ohmic heat production is increased, then the heat absorbed by mixing of entropy anomalies increases in pace to ensure that the total energy change remains the same. This conclusion is illustrated by an explicit calculation in Appendix B.

Turning now to compositional convection, gravitational energy is released by net compositional segregation into light and heavy components. We define the energy released by segregation to be that due directly to compositional redistribution, thus excluding that released by the consequent hydrostatic compression of the core as the radial mass distribution changes. Thus from (9) we infer that the gravitational energy released by segregation must appear as heat and eventually contribute to the CMB heat flux, whether the means of segregation is convective or diffusive. If the segregation occurs by convection then we argue that the heat is produced by ohmic dissipation. On the other hand, if the same segregation were to occur by diffusion, then conservation of total energy demands that the same amount of heat would be generated by the diffusive flux $\mathbf{i}$ along the gradient in chemical potential $\mu$. However, convection is actually sufficiently rapid to neglect the effects of chemical diffusion [*Gubbins et al.*, 1979; *Loper and Roberts*, 1981], and hence the gravitational energy release due to compositional segregation in (9) is equal to the compositional contribution to the ohmic dissipation in (7).

In summary, putting the effects of thermal and compositional convection together, the global heat and energy budgets (7) and (9) depend only on the net cooling of the core and the segregation of light and heavy components. The cooling may occur by either conduction or thermal convection with the total heat release being the same owing to a cancellation between the ohmic dissipation and the advective redistribution of heat, whereas the segregation occurs almost entirely by compositional convection. (In principle, the advective redistribution of heat includes heat produced by compositionally driven ohmic dissipation [*Lister and Buffett*, 1995], though we later assume that such dissipation is relatively uniform in the outer core so that redistribution is unnecessary.) We now proceed to develop a model for the thermal evolution of the core from (7) together with a physically motivated expression for the ohmic dissipation $\Phi$ in terms of the total convective energy release.

## 3. A Simple Model

### 3.1. Equations for a Well-Mixed Core

The cooling of the core is largely controlled by the relatively sluggish and more massive mantle, which regulates the total heat flux $Q(t) = \int_S \mathbf{q} \cdot \mathbf{dS}$ across the CMB. The magnitude and time dependence of $Q(t)$ depends on the properties of the overlying mantle convection, which include the distribution of radioactive isotopes and any compositional or rheological layering in the mantle. As many of the details of mantle convection are unknown, we treat $Q(t)$ as a prescribed parameter in order to focus on the evolution of the core.

We continue to assume that the outer core convects vigorously so that the temperature profile is approximately adiabatic and the composition is approximately uniform. It should be noted that, though the fluctuations about an adiabatic, hydrostatic, isentropic and well-mixed state are small, the correlation of these small fluctuations with the convective velocities is responsible for the buoyancy fluxes that sustain the dynamo.

When calculating $\Phi$, it should also be noted that transport of heat and light elements in the core occurs by both convection and diffusion along the large radial gradients in the average temperature and pressure. In particular, thermal conduction along the adiabat may represent a significant fraction, or even all, of the total heat flux. To account for this, we define the convective heat flux $\mathbf{q}^*$ to be the difference between the total heat flux from the core and the heat flux that would be conducted along the adiabat at the CMB. Thus

$$\int_S \mathbf{q}^* \cdot \mathbf{dS} = \left(\frac{Nu - 1}{Nu}\right) Q(t), \tag{10}$$

where the Nusselt number $Nu$ is the ratio of the total heat flux $Q$ to that conducted along the adiabat at the CMB. The diffusive transport of light elements down the radial pressure gradient is believed to be much slower and less important than the effects of thermal diffusion [*Gubbins et al.*, 1979; *Loper and Roberts*, 1981]. Hence, while only the convective heat flux $\mathbf{q}^*$ contributes to the thermal part of $\Phi$, all the gravitational energy released by compositional segregation contributes to the compositional part of $\Phi$. (If $Nu < 1$, then the contribution due to $\mathbf{q}^*$ is negative [*Loper*, 1978b], as discussed in section 5.)

We assume that the convective mixing of compositional and thermal anomalies is efficient, so that the associated release of gravitational energy is largely converted to motion and thence to ohmic heating before diffusion smooths out the anomalies. Under this assumption the rate of ohmic dissipation $\Phi$ is given by (see Appendix A)

$$\Phi = 4\pi c^2 \frac{dc}{dt}\left(\Delta\rho + \frac{\alpha\rho_0 L}{C_P}\right)[\overline{\psi} - \psi(c)] - \frac{\alpha}{C_P}\int_S \mathbf{q}^* \cdot \mathbf{dS}\,[\overline{\psi} - \psi(b)], \tag{11}$$

where $\Delta\rho$ is the density jump across the ICB that is due to compositional differences and not to the phase change, $\alpha$ is the coefficient of thermal expansion, $L$ is the latent heat per unit mass,

$$\overline{\psi} = \frac{1}{M_{oc}}\int_{V_{oc}} \rho\psi dV \tag{12}$$

is the mass-averaged value of the gravitational potential in the outer core, and $c$ and $b$ are the inner and outer radii of the fluid core. Numerical values for the core properties are listed in Table 1. The increment $\Phi\, dt$ can be interpreted as the energy released in time $dt$ by the

**Table 1.** Physical Properties of the Earth's Core

| Parameter | Symbol | Value |
|---|---|---|
| Latent heat | $L$ | $6.0 \times 10^5$ J kg$^{-1}$ |
| Specific heat capacity | $C_P$ | 800 J K$^{-1}$ kg$^{-1}$ |
| Thermal conductivity | $k$ | 35 W K$^{-1}$ m$^{-1}$ |
| Thermal expansivity | $\alpha$ | $10^{-5}$ K$^{-1}$ |
| Mass of core | $M_c$ | $1.95 \times 10^{24}$ kg |
| Mass of light element | $M_l$ | $6.8 \times 10^{22}$ kg |
| Density jump* | $\Delta\rho$ | 400 kg m$^{-3}$ |
| Current outer-core radius | $b$ | $3.48 \times 10^6$ m |
| Current inner-core radius | $c$ | $1.22 \times 10^6$ m |

*The nominal value of the density jump due to compositional change across the inner-core boundary. Other parameters are taken from *Dziewonski and Andersen* [1981], *Masters* [1979], *Masters and Shearer* [1990] and *Stevenson* [1981].

redistribution of an excess mass per unit area of $-(\Delta\rho+\alpha\rho_0 L/C_P)\,dc$ at the ICB and $\alpha(q^* dt/C_P)$ at the CMB throughout the outer core. Alternatively, the thermal contribution can be interpreted as the work done by the thermal buoyancy associated with the transport of heat $4\pi c^2 \dot{c}\rho_0 L\,dt$ from the CMB to the ICB and use of the remainder, $4\pi(b^2 q^* - c^2\dot{c}\rho_0 L)dt$, to cool the outer core.

We now proceed to evaluate the global heat budget (7). The contribution of compositional changes to the internal energy through heat of mixing or diffusion is roughly two orders of magnitude smaller than the leading-order terms [*Gubbins et al.*, 1979]. Thus (7) reduces to

$$\int_V \rho T \frac{Ds}{Dt} dV = -Q(t) + \int_V \frac{J^2}{\sigma}\, dV\,. \qquad (13)$$

The effects of cooling are made explicit by expanding the change in entropy on the left-hand side of (13) in terms of changes in pressure and temperature under the usual assumption of local thermodynamic equilibrium. We again neglect small effects due to changes in composition, such as the heat of mixing [*Gubbins et al.*, 1979], and obtain

$$\int_V \rho T \frac{Ds}{Dt} dV = \int_V \rho C_P \frac{DT}{Dt} dV - \int_V \alpha T \frac{DP}{Dt} dV - \int_{S_i} \rho L \frac{dc}{dt} dS\,, \qquad (14)$$

where $S_i$ is the surface of the inner core. Integration of the terms in (14) is complicated by the advective derivatives, such as $\mathbf{v}\cdot\nabla s$, in the expansion of the material derivatives. Although we assume that the core is well mixed and adiabatic on average, small deviations from the condition $\nabla s = 0$ are a necessary consequence of convection, and values of $\mathbf{v}\cdot\nabla s$ comparable to $\partial s/\partial t$ will be required locally to ensure that the average change in entropy occurs uniformly, as required to maintain a well-mixed state (see Appendix B).

The chief nonuniformities in the rate of entropy production occur in the thermal boundary layer at the CMB (when $Nu \neq 1$) and are due to latent-heat release at the ICB in response to cooling. These are redistributed by convection and, as argued in section 2.3 and illustrated by explicit calculation in Appendix B, the advective term $\int \rho T \mathbf{v}\cdot\nabla s\, dV$ in (14) cancels the thermal contribution $\Phi_T$ to the convective energy release (11) so that neither appear in the overall heat balance (13). Using this fact and replacing the partial derivative of $s$ with partial derivatives of $T$ and $P$, we may write (13) as

$$Q = -\int_V \rho C_P \frac{\partial T}{\partial t} dV + \int_V \alpha T \frac{\partial P}{\partial t} dV + \int_{S_i} \rho L \frac{dc}{dt} dS + \Phi_C\,, \qquad (15)$$

where $\Phi_C = \Phi - \Phi_T$ denotes the compositional part of the convective energy release given by (11).

The temporal changes in $T$ and $P$ may be estimated by omitting the very small fluctuations due to convection. Consequently, the changes in $T$ and $P$ represent changes in the average conditions in the core, which we take to be hydrostatic, adiabatic and compositionally uniform. The latter two conditions apply only to the fluid outer core. The averaging time is long compared with the timescale of the convective fluctuations, but short compared with the time over which the Earth cools. The average $P$, $T$ and $C$ are described by

$$\frac{\partial P}{\partial r} = -\rho g \qquad (16)$$

$$\frac{\partial T}{\partial r} = -\frac{\rho g \gamma T}{K_s} \qquad (c < r < b) \qquad (17)$$

$$\frac{\partial C}{\partial r} = 0 \qquad (c < r < b) \qquad (18)$$

where $\gamma$ is the Gruneisen parameter, $K_s$ is the adiabatic bulk modulus, the acceleration due to gravity $g$ is given by

$$g(r) = \frac{4\pi G}{r^2} \int_0^r \rho(x) x^2 dx\,, \qquad (19)$$

and the density $\rho$ is generally a function of $P$, $T$ and $C$; particular equations of state will be assumed in later sections.

Equations (16)–(19) are integrated subject to three boundary conditions. First, the surface of the inner core is assumed to be in thermodynamic equilibrium with the surrounding liquid so that $T(c,t)$ is equal to the liquidus temperature $T_L(P,C)$. Over a limited range of pressure and composition, the liquidus temperature can be approximated by a linear expansion of the form

$$T_L(P,C) = T_L(P_0,C_0) + \frac{\partial T_L}{\partial P}(P-P_0) + \frac{\partial T_L}{\partial C}(C-C_0)\,, \qquad (20)$$

where $P_0$ and $C_0$ are the pressure and concentration at the center of the Earth immediately prior to the formation of the solid inner core (see Table 2 for the parameter values which define the liquidus temperature). Second, the core liquid is modeled as an ideal mixture

**Table 2.** Parameters of the Taylor-Series Representation of the Solidification Temperature

| Parameter | Symbol | Value |
|---|---|---|
| Reference temperature | $T_L(P_0, C_0)$ | 5000 K |
| Reference composition | $C_0$ | 0.035 |
| Reference pressure | $P_0$ | $3.62 \times 10^{11}$ Pa |
| Pressure dependence | $\partial T_L/\partial P$ | $1.0 \times 10^{-8}$ K Pa$^{-1}$ |
| Compositional dependence | $\partial T_L/\partial C$ | $-1.50 \times 10^4$ K |

Parameters are based on *Boehler* [1993], *Brown and McQueen* [1986], *McQueen and Marsh* [1966] and *Svendsen et al.* [1989].

of heavy and light components such that the solid phase is composed entirely of the heavy component and the light component is present in the liquid only. Since the outer core is compositionally homogeneous, $C$ is given by

$$C(t) = \frac{M_l}{M_{oc}(t)} \qquad (c < r < b), \tag{21}$$

where $M_l$ is the constant mass of light element in the core and $M_{oc}(t)$ is the slowly decreasing mass of the outer core. Finally, the pressure is determined from the condition that $P$ vanishes at the Earth's surface.

### 3.2. Analytic Solution

In addition to the boundary conditions, the material properties $\rho$, $K_s$ and $\gamma$, which may themselves be functions of $P$, $T$ and $C$, must be prescribed. Equations (16)–(19), in their most general form, are thus nonlinear and require numerical solution. In view of the advantages of analytical solutions, we introduce a number of approximations to the form of the material properties and thereby eliminate the need for numerical integration. Specifically, we neglect the known temporal variations in $\rho$, $K_s$ and $\gamma$ and represent their radial dependence in power series. In practice, it suffices to retain only the leading-order constants $\rho_0$, $K_0$ and $\gamma_0$, which represent the values at the center of the Earth, though higher-order terms could easily be retained if required. We confirm the validity of these approximations in section 4 by comparison of our results with full numerical solutions that use more realistic estimates of the material properties.

Equations (16) and (19) become linear on replacing $\rho$, $K_s$, and $\gamma$ with constants $\rho_0$, $K_0$, and $\gamma_0$. The solutions for $P$ and $T$ are given by

$$P(r) = P_0 - Ar^2, \tag{22}$$

$$T(r,t) = T_L(c)e^{-\phi(r^2-c^2)/b^2}, \tag{23}$$

where $A = 2\pi G\rho_0^2/3$, $\phi = Ab^2\gamma_0/K_0$ and $T_L(c)$ is the liquidus temperature expressed as a function of the inner-core radius $c(t)$ by using (21) and (22) to eliminate $C$ and $P$ from (20). Direct substitution produces

$$T_L(c) = T_L(0) - \frac{\partial T_L}{\partial P}Ac^2 + \frac{\partial T_L}{\partial C}\left(\frac{c^3}{b^3 - c^3}\right)C_0, \tag{24}$$

where $C_0$ is the concentration of light elements when the core is entirely fluid, though we later eliminate $T_L(0)$ from (24) by use of the identity

$$T_L(0) = \frac{Ab^2}{\phi}\left(\frac{\partial T}{\partial P}\right)_0 \tag{25}$$

derived from (16) and (17), where $(\partial T/\partial P)_0$ is the adiabatic gradient evaluated at $P = P_0$ or, equivalently, at the center of the Earth when $c = 0$.

The solutions for $P$, $T$ and $C$ may now be used to evaluate the various terms in the global heat balance (15). From (22), $\partial P/\partial t = 0$ so that the energy balance is independent of $P$. Although this is an approximation due to the assumption of a constant density, we show in section 4 that the errors incurred are small. Thus, once the total heat flux $Q$ is prescribed, the energy balance depends only on the rates of change of temperature $T$ and inner-core radius $c$, which are related by the assumption of thermodynamic equilibrium. From (23) this relationship is given explicitly by

$$\frac{\partial T(r,t)}{\partial t} = \frac{d}{dc}\left(T_L(c)e^{\phi c^2/b^2}\right)e^{-\phi r^2/b^2}\frac{dc}{dt}, \tag{26}$$

where $T_L(c)$ is given by (24). Since the temperature variation across the core is small ($\phi = 0.26$), we use the approximation

$$e^{-\phi(r^2-c^2)/b^2} = 1 - \phi(r^2 - c^2)/b^2 + O(\phi^2) \tag{27}$$

to express the volume integral of $\partial T/\partial t$, which appears in (15), as

$$\int_V \rho_0 C_P \frac{\partial T}{\partial t} dV = H\left(\frac{dT_L(c)}{dc} + 2\frac{c\phi}{b^2}T_L(c)\right)\frac{dc}{dt}, \tag{28}$$

where

$$H = \frac{4\pi b^3 \rho_0 C_P}{3}\left[1 - \phi\left(\frac{3}{5} - \frac{c^2}{b^2}\right)\right]. \tag{29}$$

In the evaluation of the integral in (28) over the entire core we have assumed that the inner core is adiabatic, which represents an intermediate situation between the limiting cases of a perfectly conducting and perfectly insulating inner core considered by *Buffett et al.* [1992]. The differences in the thermal evolution of the core between these cases were found to be negligible until the core is almost totally solid. Adoption of an adiabatic inner core is thus a convenient and accurate approximation.

By integration of $\mathbf{g} = -\nabla\psi$ and (19), the gravitational potential is shown to be

$$\psi = Ar^2/\rho_0 + \text{const} \tag{30}$$

so that the average potential in the outer core, which is needed to evaluate (11), is given by

$$\overline{\psi} = \frac{3A}{5\rho_0}\frac{b^5 - c^5}{b^3 - c^3} + \text{const}\,. \tag{31}$$

We can now use (28) to eliminate the dependence on $T$ in the heat balance (15) in favour of a dependence on $c$ and thus obtain an equation for the growth of the inner core. Substitution of (11), (24) and (28) into (15) reduces it to an ordinary differential equation for $\eta(t) \equiv c(t)/b$ of the form

$$f(\eta)\frac{d\eta}{dt} = Q(t)\,, \tag{32}$$

where the heat flux $Q(t)$ appears as a forcing term. (The algebra is straightforward but messy.) Each of the terms comprising $f(\eta)$ takes the form of a simple rational function of $\eta$, and hence an analytical integral for $\eta(t)$ may be obtained once $Q(t)$ is prescribed. The detailed algebraic form of the result is not particularly illuminating, though we can obtain useful approximate results by assuming that $\eta$ is sufficiently small to neglect terms in $f(\eta)$ of order $\eta^3$ and smaller. The accuracy of this approximation may be assessed by direct numerical integration of (32), and for the current value of $\eta$ we find that the error incurred is less than 2%. On dropping the higher-order terms we integrate (32) to obtain

$$\mathcal{M}^{-1}\int_0^t Q(\tau)d\tau = \eta^2 + (\mathcal{G}_C + \mathcal{L} - \mathcal{C})\,\eta^3 \;+\; O(\eta^4)\,, \tag{33}$$

where time is measured from the instant at which the temperature first fell below the liquidus at the center of the Earth and the inner core began to grow. The cumulative heat flux from the core is made dimensionless in (33) by the factor

$$\mathcal{M} = 4\pi\left(\frac{1}{3} - \frac{\phi}{5}\right)\rho_0 C_P A b^5\left[\frac{\partial T_L}{\partial P} - \left(\frac{\partial T}{\partial P}\right)_0\right]\,, \tag{34}$$

which represents the heat that must be extracted to cool the entire core to its solidification temperature. This choice of scale makes the total dimensionless heat released due to cooling unity. The other parameters in (33) are the dimensionless numbers

$$\begin{aligned} \mathcal{C} &= \frac{C_0}{Ab^2}\frac{\partial T_L}{\partial C}\left[\frac{\partial T_L}{\partial P} - \left(\frac{\partial T}{\partial P}\right)_0\right]^{-1}, \\ \mathcal{G}_C &= \frac{4\pi}{5}\frac{Ab^5}{\mathcal{M}}\frac{\Delta\rho}{\rho_0}, \\ \mathcal{L} &= \frac{4\pi}{3}\frac{\rho_0 L b^3}{\mathcal{M}}, \end{aligned} \tag{35}$$

which describe the relative importance of the various physical processes in the evolution of the core. Specifically, $\mathcal{C}$ reflects the effect of composition on the liquidus temperature, $\mathcal{L}$ represents the effect of latent-heat release, and $\mathcal{G}_C$ represents the effects of gravitational energy release and ohmic dissipation due to compositional rearrangement.

### 3.3. Results

The analytic solution (33) for the growth of the inner core provides a useful starting point for an investigation of the thermal evolution of the core and the energetics of the dynamo. In addition to quantitative results, which can be calculated once the numerical parameters in (34) and (35) have been estimated, the solution offers insight into the processes that dominate the growth of the inner core. For example, when the inner core is small (i.e., $\eta << 1$), the heat due to cooling controls its growth. As the inner-core radius increases, the release of both latent heat and gravitational energy due to compositional rearrangement becomes more important. For the present state $\eta = 0.35$, we find that the rate of growth is still largely controlled by the heat due to cooling. We can exploit this result to obtain an explicit expression for $\eta(t)$ in the form

$$\eta(t) = \eta_0(t) - \delta\eta(t)\,, \tag{36}$$

where the leading-order term is

$$\eta_0(t) = \left[\mathcal{M}^{-1}\int_0^t Q(\tau)d\tau\right]^{1/2} \tag{37}$$

and the correction term

$$\delta\eta(t) = \frac{(\mathcal{G}_C + \mathcal{L} - \mathcal{C})\,\eta_0^2}{2 + 3\,(\mathcal{G}_C + \mathcal{L} - \mathcal{C})\,\eta_0} + O(\eta_0^4) \tag{38}$$

represents the smaller effects of latent heat and compositional rearrangement which are associated with the parameters $\mathcal{L}$, $\mathcal{G}_C$ and $\mathcal{C}$.

Numerical values for the model parameters are obtained using the physical properties listed in Tables 1 and 2. Many of the physical properties are poorly known, and as a result, the model parameters listed in Table 3 are poorly constrained. The most serious uncertainty occurs in $\mathcal{M}$, which depends on the difference $\partial T_L/\partial P - (\partial T/\partial P)_0$. The difference in the gradients may be expressed in a form more convenient for estimation by adopting Lindemann's law to describe the melting curve [e.g., *Stacey*, 1992],

$$\frac{dT_L}{dP} = 2\left(\gamma - \frac{1}{3}\right)\frac{T_L}{K_T}\,, \tag{39}$$

where $K_T$ is the isothermal bulk modulus, and by noting that the adiabatic gradient may be written as $\gamma T/K_s$. Thus the difference in gradients at the center of the Earth is

$$\frac{dT_L}{dP} - \left(\frac{dT}{dP}\right)_0 = \frac{T_L(0)}{K_T}\left[\left(2 - \frac{K_T}{K_s}\right)\gamma - \frac{2}{3}\right]\,. \tag{40}$$

**Table 3.** Model Parameters for the Analytical Solution

| Parameter | Value |
|---|---|
| $\rho_0$ | $1.23 \times 10^4$ kg m$^{-3}$ |
| $K_0$ | $1.40 \times 10^{12}$ Pa |
| $\gamma_0$ | 1.4 |
| $A$ | 0.0211 Pa m$^{-2}$ |
| $\phi$ | 0.256 |
| $\mathcal{M}$ | $1.88 \times 10^{30}$ J |
| $\mathcal{C}$ | $-0.41$ |
| $\mathcal{G}_C$ | 0.47 |
| $\mathcal{G}_T$ | 0.10 |
| $\mathcal{L}$ | 0.69 |

A similar approach was used by *Loper* [1991], who also took account of the effects of composition. Each parameter in (40) is subject to some uncertainty, but the largest uncertainty is probably due to $T$ near the center of the Earth, estimates of which vary from 4000 K to 6000 K [e.g., *Brown and McQueen*, 1986; *Williams et al.*, 1987; *Boehler*, 1993]. We can expect at least a comparable level of uncertainty in $\mathcal{M}$.

Specific predictions for the growth of the inner core also require estimates of $Q(t)$. *Stacey* [1992, p. 301] estimates that the Earth is currently cooling at a net rate $10^{13}$ W, which represents approximately one-quarter of the heat flux measured at the Earth's surface [*Sclater et al.*, 1980]. A large part of the $10^{13}$ W is attributed by *Stacey* [1992] to the cooling of the mantle; the rest of the cooling is partitioned between the crust and the core. His preferred estimate of the current heat loss $Q$ from the core is $3.0 \times 10^{12}$ W, which agrees well with an estimate obtained by *Sleep* [1990] on the basis of the heat transported by mantle plumes. Some coupled calculations of the cooling of the mantle and core [e.g., *Stevenson et al.*, 1983; *Mollett*, 1984] suggest that the heat flux from the core may have decreased slowly with time, through an amount that depends on the initial conditions of the Earth, which are poorly known. Accordingly, we consider $Q = 4.0 \times 10^{12}$ W as a typical time-averaged value to illustrate the use of our model, but we also consider $Q = 6.0 \times 10^{12}$ W and $Q = 2.5 \times 10^{12}$ W as high and low values to span a plausible range of solutions. The probable variation of $Q$ with time could easily be included in the solution, although this level of detail seems unwarranted given the present uncertainty in the time-averaged value. We do not consider the existence of the geomagnetic field for at least $3.5 \times 10^9$ years [*McElhinny and Senanayake*, 1980] to be a strong constraint either on the age of the inner core, since the early dynamo might have been thermal, or on the variation of $Q$, due to uncertainties in the relationship between estimated paleointensity and predicted ohmic dissipation. The low value of $Q$ was chosen to allow growth of the inner core to its present radius within the approximate age of the Earth. This value is actually less than the heat flux conducted along the adiabatic gradient (i.e., $2.8 \times 10^{12}$ W), a possibility which was first noted by *Loper* [1978a,b]. If the core is well mixed, then $2.8 \times 10^{12}$ W will still be conducted along the adiabat, and thus the smaller net heat flux across the CMB requires a downward convective transport of heat into the interior of the core, driven by compositional convection.

Figure 1 shows the growth histories calculated using the extreme and illustrative values of $Q$. For the illustrative value $Q = 4.0 \times 10^{12}$ W, the inner core grows to its present radius in $2.8 \times 10^9$ years, which is somewhat longer than the values obtained by *Stevenson et al.* [1983]. The values calculated using the extreme values of $Q$ are $1.9 \times 10^9$ and $4.6 \times 10^9$ years. The ages span a wide interval, indicating that plausible variations in $Q$ can lead to significant differences in the predicted age of the present-day inner core. The corresponding growth rates are closely linked to the energy supply for the geodynamo, so the value of $Q$, which is related to the strength of convection in the mantle, is extremely important. The growth rate is also dependent on the effect of the slowly changing composition on the liquidus temperature. The dashed line in Figure 1 shows the growth history calculated using the illustrative value $Q = 4.0 \times 10^{12}$ W and $\partial T_L/\partial C = 0$. Since the liquidus temperature is depressed by increasing values of $C$, the neglect of $\partial T_L/\partial C$ in this calculation reduces the amount of heat that must be extracted to solidify the inner core, and as a result, the inner core grows more rapidly for a given value of $Q$. Differences between the solutions in Figure 1 indicate that both the phase diagram and the heat flux $Q$ are important inputs for the calculation of the growth of the inner core. Unfortunately, both are poorly known. The advantage

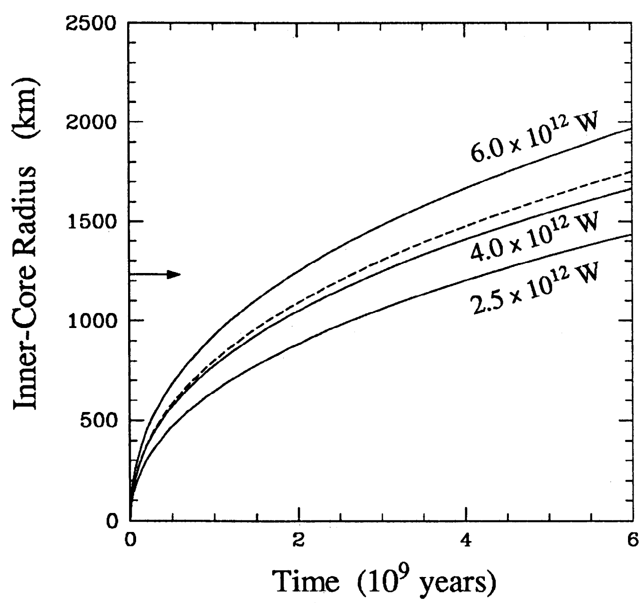


**Figure 1.** Radius of the inner core measured from the time when solidification begins. Solutions are presented for three estimates for the net heat flux across the core-mantle boundary (shown in units of $10^{12}$ W). The dotted line corresponds to a heat flux $4.0 \times 10^{12}$ W and $\partial T_L/\partial C = 0$. The arrow indicates the current radius of the inner core.

of our analytical formulation is that the consequences of changes in the input parameters can easily be determined if improved estimates become available.

The importance of the various heat sources may be assessed from the simple form of the dimensionless heat budget in (33). The heat sources may be identified with terms that appear on the right-hand side. For example, the heat due to cooling is represented by the term $\eta^2 - \mathcal{C}\eta^3$, whereas the latent-heat release and the ohmic dissipation due to compositional convection are represented by $\eta^3\mathcal{L}$ and $\eta^3\mathcal{G}_C$, respectively. By differentiating (33) with respect to time, we obtain estimates of the heat sources that contribute to the heat flux across the CMB. Figure 2 shows the result of a calculation using the illustrative value of heat flux $Q = 4.0 \times 10^{12}$ W. The heat due to cooling, denoted by curve S, dominates the other heat sources until the core is almost completely solidified [cf. *Gubbins et al.*, 1979; *Loper*, 1991].

## 4. Numerical Model

We now assess the validity of the simplifications that were used above to obtain an analytical solution by adopting a more general equation of state which is a function of $P$, $T$ and $C$. We also allow the volume of the Earth and the pressure at the center to evolve under the influence of self-gravitation. The pressure dependence of $\rho$ is based on a model advocated by *Loper* [1978a], in which the bulk modulus is a linear function of pressure, but is independent of temperature and composition. Consequently, we express $K_s(P)$ as

$$K_s(P) = (P + P_n)/a_n\,, \qquad (41)$$

where $P_n$ and $a_n$ are parameters chosen to fit the current seismic profiles of $K_s$ in the various regions of the

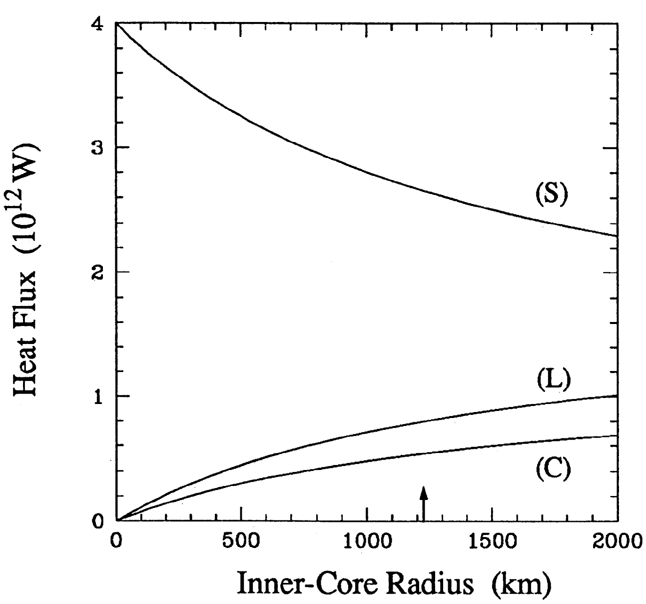


**Figure 2.** Contribution of various heat sources to the heat flux across the core-mantle boundary for $Q = 4.0 \times 10^{12}$ W. The largest heat source is secular cooling (curve S). The remaining curves are latent heat (L) and ohmic heat due to compositional convection (C).

Earth [e.g., *Dziewonski and Anderson*, 1981]. Here we consider five regions. The core parameters are denoted by a subscript $n = 1$, while the subscripts $n = 2, 3, 4, 5$ pertain to the lower mantle, the transition zone, the upper mantle and the crust, respectively. The parameter values used here are listed in Table 4.

The bulk modulus is defined by

$$K = \rho \frac{dP}{d\rho} \qquad (42)$$

so that the pressure dependence of the density field is obtained by integration, yielding

$$\rho = \rho_n^0(T, C)(1 + P/P_n)^{a_n}\,, \qquad (43)$$

where $\rho_n^0$ is the density at zero pressure. The thermal and compositional dependence of $\rho$ in the core is taken as

$$\rho_1^0(T, C) = \rho_1^0(0, C)e^{-\alpha_1 T}\,, \qquad (44)$$

$$1/\rho_1^0(0, C) = (1 - C)/{\rho_h}^0 + C/{\rho_l}^0\,, \qquad (45)$$

where $\rho_h^0$ and $\rho_l^0$ are the densities of the heavy and light elements in the core at zero pressure and temperature. We chose values for $\rho_h^0$ and $\rho_l^0$ such that our model yields in situ estimates of $\rho$ which agree with estimates of the current Earth structure [e.g., *Dziewonski and Anderson*, 1981] when the inner-core radius reaches its current value. The model yields a value for the present-day density jump across the ICB of approximately 600 kg $m^{-3}$, of which we attribute 200 kg $m^{-3}$ to the phase change and 400 kg $m^{-3}$ to the compositional variation (see Table 1). The density of the mantle has a temperature dependence given by (41), but is independent of $C$.

Equations (16)–(18) for pressure, temperature, and composition were integrated subject to the three conditions

$$\begin{aligned} P(a) &= 0\,, \\ T(c) &= T_L(c)\,, \\ C(r) &= M_l/M_{oc} \quad (c < r < b)\,, \end{aligned} \qquad (46)$$

where $a$ is the radius of the Earth's surface. In order to evaluate $P(a)$, the hydrostatic equation (16) must be integrated to the Earth's surface. Since the density of the mantle depends on temperature, we must also determine the temperature in the mantle. For this purpose we assumed that the mantle is adiabatic everywhere except in the thermal boundary layers. The boundary-layer regions were modeled by introducing discontinuities into the adiabatic temperature. Fortunately, the calculations proved to be insensitive to the details of the temperature profile in the mantle, so the assumptions made about the thermal boundary layers are not crucial to the present analysis. For example, in the calculations that follow we assumed a temperature jump of 1000 K at the CMB, but increasing or decreasing this value by 50% produced virtually no change in the resulting growth of the inner core.

Numerical solutions of (16)–(18) were calculated at predetermined values of $c$, so that the resulting fields $P$,

**Table 4.** Parameters of Compressible Earth Model

| Parameter | Zone | | | | |
|---|---|---|---|---|---|
| | $n = 1$ | $n = 2$ | $n = 3$ | $n = 4$ | $n = 5$ |
| $\gamma_n$ | 1.4 | 1.2 | 1.2 | 1.3 | 1.3 |
| $\alpha_n$ ($10^{-5}$ K$^{-1}$) | 1.0 | 3.0 | 3.0 | 3.0 | 3.0 |
| $P_n$ ($10^{10}$ Pa) | — | 6.80 | 1.13 | 1.40 | 0.0 |
| $a_n$ | 0.304 | 0.304 | 0.131 | 0.140 | 0.00 |
| $\rho_n^0$ ($10^3$ kg m$^{-3}$)* | — | 4.30 | 3.67 | 3.38 | 2.87 |
| Interface pressure† | — | 2.40 | 1.40 | 0.05 | 0.00 |

*The density of the core at zero pressure and temperature is determined from (45) using the values $\rho_h^0 = 7.8 \times 10^3$ kg m$^{-3}$ and $\rho_l^0 = 3.3 \times 10^3$ kg m$^{-3}$.

†The top of the $n$th zone is defined by a constant pressure surface in units of $10^{10}$ Pa. The radii of the core-mantle boundary and of the Earth's surface are set by conservation of mass in their respective volumes. The mass of the Earth is $5.97 \times 10^{25}$ kg, while the mass of the core is given in Table 1.

$T$ and $C$ were known as a function of $c$. The solutions were determined simultaneously with the positions of several boundaries which are unknown prior to the calculation. In particular, the positions of the CMB, the Earth's surface, and the upper and lower boundaries of the mantle transition zone were not known precisely in advance of the calculation. The positions of the CMB and the Earth's surface were determined by imposing conservation of mass on both the core and the Earth as a whole. The boundaries of the transition zone were determined by pressure and temperature conditions which define the positions of the two major phase transitions within the mantle. To avoid having to model mantle temperatures precisely, we fixed the location of the phase boundaries using the pressure only. The upper and lower boundaries of the transition zone were defined by $P = 1.4 \times 10^{10}$ Pa and $P = 2.4 \times 10^{10}$ Pa, respectively. The numerical scheme used to solve (16)–(18) is an iterative procedure. The equations were solved by a shooting method using approximate boundary locations. This solution was used to adjust the positions of the boundaries, and the calculation was repeated until convergence was achieved. Three or four iterations were typically required.

Using numerically determined profiles of $P$ and $T$ at successive values of $c$, the time derivatives of $P$ and $T$ in (15) were evaluated from

$$\int_V \rho T \frac{\partial s}{\partial t} dV =$$

$$\left[\int_V \rho C_P \frac{\partial T}{\partial c} dV - \int_V \alpha T \frac{\partial P}{\partial c} dV - \int_{S_i} \rho L dS\right] \frac{dc}{dt}. \quad (47)$$

The partial derivatives of $P$ and $T$ with respect to $c$ were approximated using first-order differences between numerical solutions at successive values of $c$. Profiles of these first-order differences were fitted using a cubic spline and the integrals in (47) were evaluated using the spline representations. The result was substituted into the energy equation to define a first-order differential equation for $c(t)$, as was done previously in our analytical solution. This differential equation was then integrated numerically using a prescribed value of the heat flux.

The numerical result for the growth of the inner core using the illustrative heat flux, $Q = 4.0 \times 10^{12}$ W, is shown in Figure 3 together with the result from our analytical model for comparison. The agreement is good given the simple approximations used for the material properties in our analytical model. In the numerical model the inner core grows to its present radius in $2.6 \times 10^9$ years, which deviates from the results of our analytical model by only 8%. This error is actually less than the known radial variations in density which were neglected in the analytical solution of section 3. The good agreement is due in part to competing effects of radial and temporal variations in the material properties. For example, the solution is sensitive to the density and bulk modulus near the ICB through their effect on the adiabatic temperature gradient. The density and bulk modulus decrease with radius at any time, but, as the inner core grows, the associated increase in pressure due to self-gravitation causes both the density and the bulk modulus to increase with time at any radius. Thus the temporal and radial variations tend to cancel at the outward-moving ICB. The heat source in (47) associated with the change in pressure is also found to be small, typically accounting for less than 1% of the heat sources in the heat balance in agreement with calculations by *Gubbins et al.* [1979]. Consequently, the neglect of this term in the analytical model is justified given the uncertainty of other effects, such as the heat flux $Q$ and the phase diagram for the liquid core.

## 5. Relative Importance of Thermal and Compositional Convection

The global heat equation (13) describes the sources of energy that contribute to the net heat flux across the CMB, but it does not quantify how these energy sources

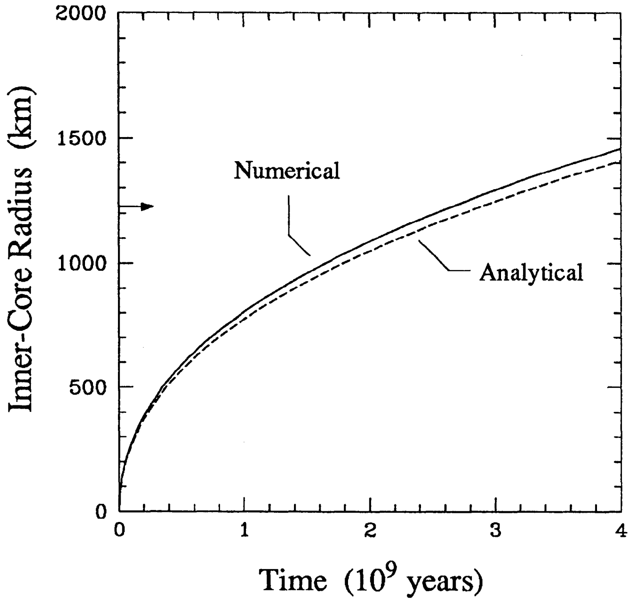


**Figure 3.** Comparison of the numerical (solid curve) and analytical (dashed curve) solutions for an average heat flux $Q = 4.0 \times 10^{12}$ W. The discrepancy in the time predicted for the inner core to grow to its present radius $c = 1221$ km is only about 8%.

contribute to the geodynamo. For example, we predict that the largest source of energy in the global heat equation is at present the heat due to cooling, but this does not imply that this heat is primarily responsible for sustaining the geodynamo. Indeed, as argued in section 2 and shown explicitly by (8), the rate of ohmic heating $\Phi$, which indicates the strength of a quasi-steady dynamo process, is given by the energy released by thermal and compositional convection.

Substituting from (30) and (31) for $\psi$ and using the nondimensionalization of our analytical model, we rewrite (11) as

$$\Phi = \mathcal{G}_T \left(\frac{Nu-1}{Nu}\right)\left(\frac{2-5\eta^2+3\eta^5}{2(1-\eta^3)}\right) Q + (\mathcal{G}_C + \tfrac{3}{2}\mathcal{G}_T\mathcal{L})\eta^2\dot{\eta}\left(\frac{3-5\eta^2+2\eta^5}{1-\eta^3}\right)\mathcal{M}\,, \tag{48}$$

where

$$\mathcal{G}_T = \frac{2}{5}\frac{\alpha A b^2}{\rho_0 C_P}\,. \tag{49}$$

We define the efficiency of convection $\epsilon$ to be the ratio of $\Phi$ to the imposed total heat flux $Q$. Thus, using (33) to relate $\dot{\eta}$ to $Q$, we find that

$$\Phi = (\epsilon_C + \epsilon_T)\int_S \mathbf{q}\cdot d\mathbf{S}\,, \tag{50}$$

where the efficiencies of the compositional and thermal convection are given by

$$\epsilon_C = \frac{3\eta\mathcal{G}_C}{2+3\eta(\mathcal{G}_C+\mathcal{L}-\mathcal{C})} + O(\eta^3)\,, \tag{51}$$

$$\epsilon_T = \left(\frac{Nu-1}{Nu} + \frac{9\mathcal{L}\eta}{4}\right)\mathcal{G}_T + O(\eta^2) \tag{52}$$

and the higher-order terms in $\eta$ have again been neglected. The two terms in (52) arise from the thermal buoyancy flux required to cool the core at a nonadiabatic rate and that required to remove the latent heat. The cooling term will dominate when the inner core is small, and the latent-heat term will dominate when the cooling rate is close to adiabatic.

Figure 4 shows the efficiencies calculated using the illustrative value $Q = 4.0 \times 10^{12}$ W. The efficiency of both thermal and compositional convection increases as the inner core grows, though that of thermal convection does so somewhat less rapidly. For the present state (i.e., $c = 1221$ km), (50)–(52) predict that the efficiencies of thermal and compositional convection are 0.088 and 0.135, respectively, while the thermal and compositional contributions to the ohmic heating are $3.5 \times 10^{11}$ W and $5.4 \times 10^{11}$ W. Of the thermal convection, roughly one third is due to superadiabatic cooling, and two thirds are due to latent-heat release. We relate the ohmic dissipation to the magnetic field strength by assuming information about the spatial distribution of the field. *Gubbins et al.* [1979] used the kinematic dynamo of *Kumar and Roberts* [1975] to show that $5.0 \times 10^{11}$ W are dissipated by a field with an average amplitude of approximately 240 G. If the spatial structure of this field is appropriate for the core, then we infer

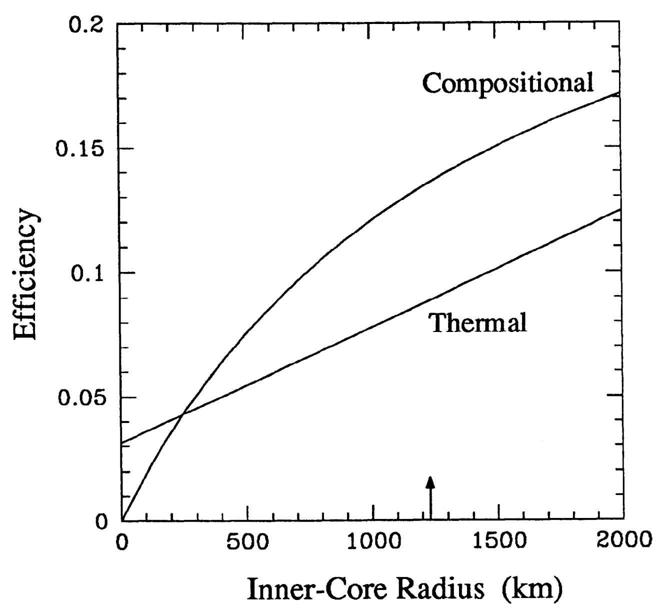


**Figure 4.** Comparison of the efficiencies of thermal and compositional convection for $Q = 4.0 \times 10^{12}$ W. The efficiencies are defined by the ratio of the ohmic heating to the heat flux across the core-mantle boundary. The efficiency of both thermal and compositional convection increases as the inner core grows though that of thermal convection does so more slowly.

that thermal convection alone is able to sustain a field of approximately 200 G, whereas the combined effects of thermal and compositional convection will produce a field of 320 G. If the inner core had not yet nucleated, then the thermal convection due solely to this rate of superadiabatic cooling would have an efficiency of 0.031 and could sustain a magnetic field of 120 G.

Thermal convection makes a larger contribution to the geodynamo with the high value of heat flux $Q = 6.0 \times 10^{12}$ W. For the present inner-core radius, the thermal and compositional contributions to the ohmic dissipation are $6.7 \times 10^{11}$ W and $8.1 \times 10^{11}$ W, respectively; thermal convection sustains a magnetic field of approximately 280 G, while the addition of compositional convection yields a total field of 415 G. Smaller fields are produced if the heat flux $Q$ has dropped below the illustrative value. For example, when $c = 1221$ km the estimated present-day heat flux $Q = 3.0 \times 10^{12}$ W [*Sleep*, 1990; *Stacey*, 1992] gives thermal and compositional contributions to the ohmic dissipation of $1.9 \times 10^{11}$ W and $4.1 \times 10^{11}$ W, which can jointly sustain a field of 260 G. The thermal efficiency in this case is due almost entirely to latent-heat release because this estimate of $Q$ is only slightly in excess of the $2.8 \times 10^{12}$ W which is conducted down the adiabatic gradient. Figure 5 shows the total efficiency $\epsilon_T + \epsilon_C$ for the high, low and illustrative values of $Q$ as a function of the inner-core radius.

For the low value of $Q$ the heat flux is less than that conducted down the adiabat at the CMB. This requires a downward convective transport of heat from the CMB into the outer core driven by compositional convection [*Loper*, 1978b] and the release of latent heat. The compositional convection may carry hot buoyant material down into the core, but the work done against gravity expends energy which would otherwise power the geodynamo. The negative efficiency for small inner-core radii indicates that there is insufficient release of compositional buoyancy and latent heat to keep the core well mixed at very early times and a stagnant zone may develop near the CMB. Thus an underlying assumption of the model breaks down at early times for low heat flux. Energy may still be supplied to the dynamo in a poorly mixed core, but these calculations suggest that the dynamo will be much less efficient. The efficiency of convection with $Q = 2.5 \times 10^{12}$ W and $c = 1221$ km is 0.18 and the associated rate of ohmic heating $4.5 \times 10^{11}$ W. The corresponding magnetic field is approximately 230 G.

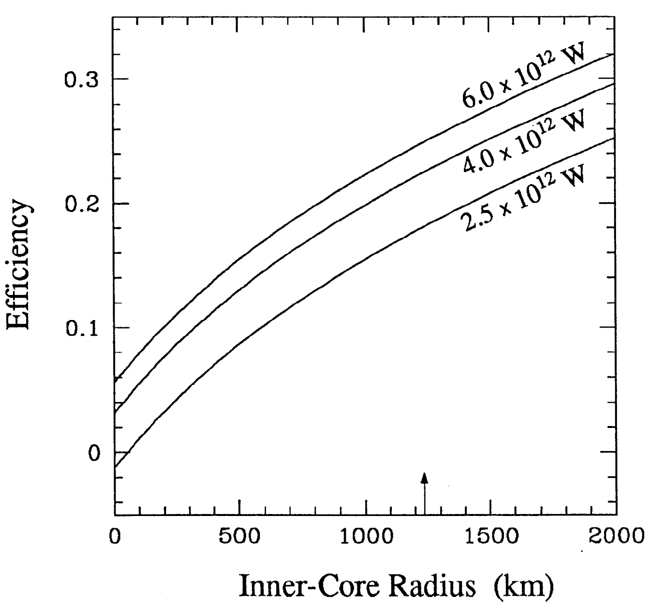


**Figure 5.** The combined efficiency of thermal and compositional convection calculated using three values of heat flux across the core-mantle boundary. The arrow indicates the current inner-core radius. The negative efficiency that occurs for the heat flux $2.5 \times 10^{12}$ W indicates that convection is too weak to sustain a well-mixed core at very early times.

The preceding estimates of the magnetic field are included for illustrative purposes only. Such estimates depend on the spatial structure of the magnetic field which is almost entirely unknown. In addition, we have assumed that the structure of the field remains constant as the energy input changes. It is equally plausible that changes in the structure of the field account for the changes in ohmic dissipation without significantly altering the amplitude of the magnetic field.

An important general conclusion to be drawn from this calculation is that the operation of the dynamo is tied to the heat flux $Q$, which is controlled by the rate of convection in the mantle. We find that a thermal dynamo is viable for a plausible range of heat fluxes. Conversely, we find that sufficiently low values of the heat flux can significantly reduce the total efficiency from the value given by compositional convection alone. These points are relevant in the context of Venus where the absence of a magnetic field has been attributed to the absence of an inner core [*Stevenson et al.*, 1983]. Our model indicates that the presence of an inner core, and the attendant generation of compositional buoyancy, is not required to sustain the magnetic field if some minimum level of heat flux is maintained. Thus weak convection in the mantle of Venus may provide an alternative or partial explanation for the absence of a field on Venus.

## 6. Conclusions

We have developed two related models of the thermal evolution of the core in order to evaluate the relative importance of thermal and compositional convection to the dynamo problem. Our analytical model permits the evolution of the inner-core radius to be expressed in a simple algebraic form and identifies the processes that play a significant role in the thermal evolution of the core. For a plausible set of input parameters we find that the heat due to cooling is the principal heat source until the core is almost completely solidified. The latent-heat release and the ohmic heat due to compositional convection have comparable effects on the thermal evolution and become increasingly important with time. The ohmic heat generated by thermal convec-

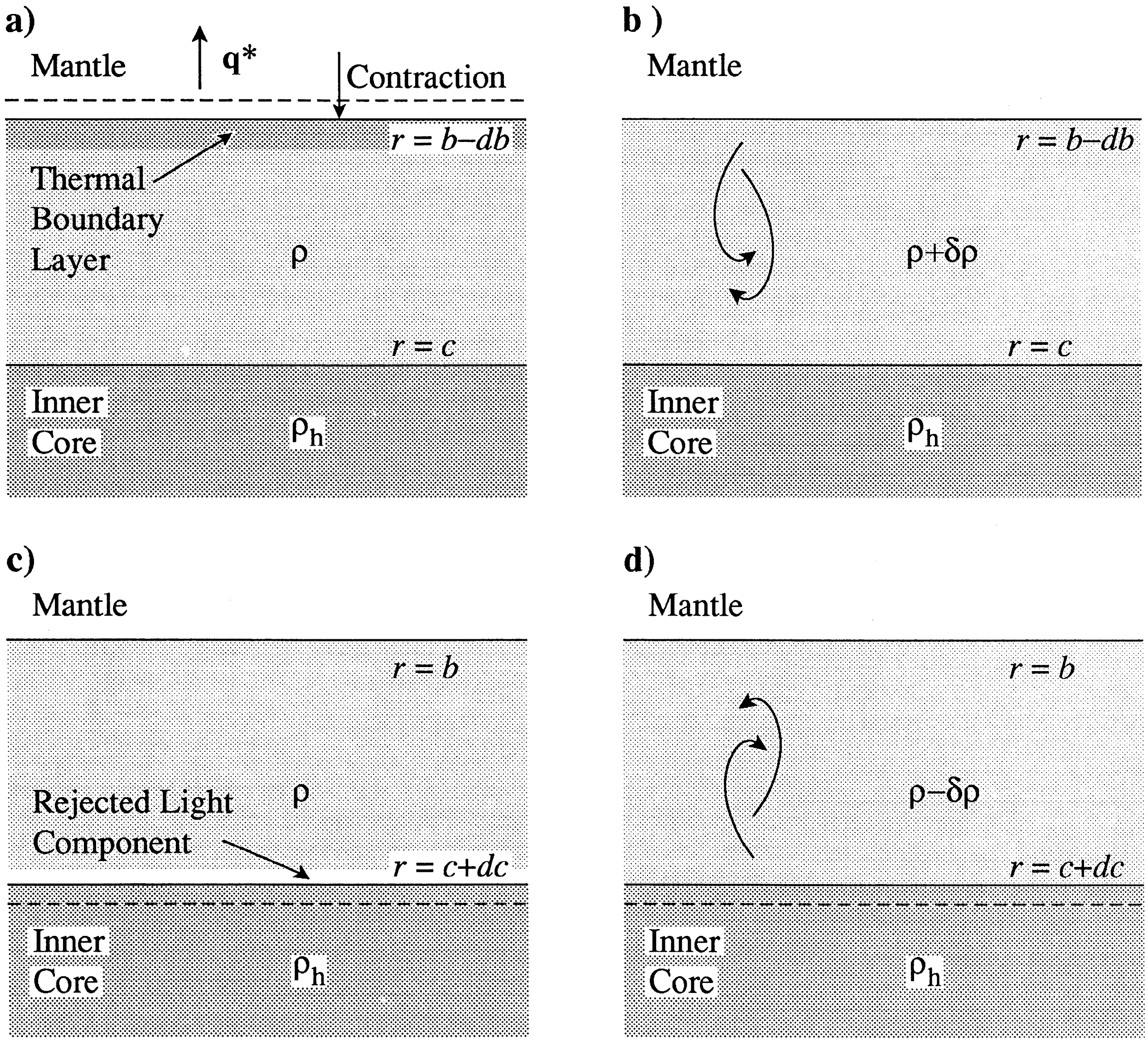


**Figure 6.** Schematic processes leading to thermal and compositional release of gravitational energy (excluding adiabatic conduction and volume changes on phase transition). (a) Heat flux $q^*$ extracted from the core causes local contraction near the core-mantle boundary. (b) The cold, dense thermal boundary layer becomes unstable and mixes into the underlying fluid. (c) Light element is excluded from the inner core as the inner core grows by solidification. (d) The light compositional boundary layer becomes unstable and mixes into the overlying fluid.

tion plays no role in the thermal evolution but makes a important contribution to the energy budget of the geodynamo, particularly at early times. We obtain explicit expressions for the efficiency of thermal and compositional convection, which indicate that both sources of convection are sufficient to sustain the magnetic field independently, although the thermally driven dynamo depends critically on heat flux across the CMB. The viability of a compositionally driven dynamo also depends on the heat flux, and a heat flux below that conducted along the adiabatic gradient can significantly reduce the total efficiency.

Our numerical model was used to test the accuracy of the assumptions invoked to develop the analytical model. We adopted a more general equation of state, based on the model of *Loper* [1978a], but with allowance for the effects of thermal contraction. Although the numerical model was more sophisticated than the analytical model, the differences between the two set of results was negligible given the present uncertainties in the thermodynamic properties of the Earth.

This work represents two significant departures from previous studies. The first is our analytical formulation which allows the consequences of changes in the input parameters to be readily assessed. This aspect of the solution is important because many of the input parameters are poorly known and are likely to change as a result of further studies. The second contribution is that we include the gravitational energy released by thermal convection and find, in contrast to previous studies, that it can make a significant contribution to the energy budget of the dynamo.

Our calculations also indicate that the relative importance of thermal and compositional convection in the dynamo problem is primarily determined by the heat flux from the core and by the inner-core radius. For an estimated current heat flux of $Q = 3.0 \times 10^{12}$ W and the current inner-core radius we find that roughly two thirds of the ohmic dissipation in the core is due to compositional convection. Since this heat flux is very close to the adiabatic value, thermal convection is relatively weak and largely due to latent-heat release, and the present-day dynamo is predominantly compositional. In the early Earth, when the inner core was smaller and the heat flux probably greater than the present values, thermal convection would have been the dominant energy source for the dynamo. Since the heat flux from the core is largely controlled by the convective transport of heat in the mantle, the absence of a magnetic field on Venus may be strongly influenced by the style of convection in the mantle of Venus. In particular, the apparent absence of a surface recycling process similar to plate tectonics on Earth suggests that there is a relatively smaller heat flux across the CMB, and hence the Venusian core may be unable to power a dynamo.

## Appendix A: Release of Gravitational Energy

In order to model the thermal evolution of the core in section 3, we required an estimate (11) of the rate $\Phi$ of conversion of gravitational energy to heat by convection and ohmic dissipation. Here we derive (11) from the assumption that vigorous thermal and compositional convection keep the outer core well mixed and nearly adiabatic. Convection at high Rayleigh numbers is characterized by boundary-layer instabilities which release gravitational energy and drive convective flow. The estimate of $\Phi$ is obtained by considering the gravitational energy released when an unstable boundary layer is mixed into the interior of the core. An alternative, more general derivation of $\Phi$ from direct consideration of the correlation between convective velocities and density fluctuations is given by *Lister and Buffett* [1995]. (See also *Braginsky and Roberts* [1995] for a related calculation.)

As is customary in thermodynamic arguments, the concurrent processes of heat extraction, convection and solidification can be divided into a number of distinct sequential steps in order to calculate the energy balances more readily. We consider the steps in three groups which correspond to general hydrostatic contraction of the core, local formation of boundary layers and convective mixing of the boundary layers.

First, general radial hydrostatic contraction of the core results from the conductive heat flux $\mathbf{q} - \mathbf{q}^*$ into the mantle and from changes in total volume associated with the phase transition at the ICB. (Gravitational energy is also released hydrostatically by barodiffusion of light element, though this is probably negligible.) Since these processes occur hydrostatically, they convert gravitational energy directly to compressional energy and a small amount of compressional (adiabatic) heating, and do not contribute to the ohmic dissipation. Second, the remaining heat flux $\mathbf{q}^*$ (if positive) produces a cold boundary layer of negatively buoyant fluid at the CMB, while exclusion of light element from the solidifying inner core produces a compositionally rich boundary layer of positively buoyant fluid at the ICB (Figures A1a and A1c). Solidification also releases latent heat [*Verhoogen*, 1961], which can be considered to contribute to the buoyancy of the layer at the ICB. Each of these processes also occurs hydrostatically, and any small radial motion associated with them again simply converts gravitational energy to compressional energy. Finally, the boundary layers become dynamically unstable and drive thermal and compositional convection in which parcels of light fluid rise and dense fluid fall through the outer core resisted by Lorentz forces (Figures A1b and A1d). We assume that convection is sufficiently vigorous that the thermal and compositional anomalies associated with fluid from the boundary layers are well mixed through the outer core before diffusion smooths them out. Thus in this final nonhydrostatic stage all the gravitational energy released by the redistribution of buoyancy from the boundary layers is converted to motion and ohmic dissipation, and there is no net pressure work.

The rate of ohmic dissipation $\Phi$ is most easily estimated from

$$\Phi = -\int_V \rho \mathbf{v} \cdot \nabla \psi \, dV = - \int_V \left(\frac{\partial \rho}{\partial t}\right) \psi \, dV - \int_S \rho \psi \mathbf{v} \cdot d\mathbf{S}, \qquad \text{(A1)}$$

where $\partial \rho / \partial t$ and $\mathbf{v}$ here describe the density change and motion associated solely with convective rearrangement (Figures 6b and 6d) and exclude the changes associated with the hydrostatic processes (Figures 6a and 6c). Thus $\mathbf{v} = 0$ on $S$ and the surface integral on the right-hand side of (A1) vanishes. The volume integral involving $\partial \rho / \partial t$ comprises both a volumetric term associated with the mean evolution of the bulk fluid and a surface term associated with the detachment of the boundary layers.

It is convenient to deal with the thermal and compositional components of gravitational energy release separately. From the equation of state (44) and the local heat balance during the boundary-layer formation, the excess mass in the dense thermal boundary layer at the CMB prior to rearrangement is $(\alpha Q^*/C_P)dt$, while the mass deficit at the ICB due to latent-heat release is $(4\pi \alpha c^2 \dot{c} \rho_0 L / C_P)dt$. The surface term in the volume integral involving $\partial \rho / \partial t$ in (A1) that corresponds to the redistribution of these mass anomalies is

$$\frac{\alpha}{C_P}\left(Q^*\psi(b) - 4\pi c^2 \dot{c}\rho_0 L\psi(c)\right) dt\,, \tag{A2}$$

where

$$Q^* = \int_S \mathbf{q}^* \cdot d\mathbf{S}\,. \tag{A3}$$

and, for simplicity, $\alpha$ is taken as uniform in the outer core. Note that although the present thermodynamic argument considers latent-heat release to occur before convective overturn, it actually occurs in response to conductive and convective cooling from above. However, this makes no difference to the net release of gravitational energy as calculated below.

The convective temperature change $dT^*$ that contributes to the bulk density change in (A1) is due to the net effect of the convective heat flux $\mathbf{q}^*$ and the latent-heat release $\rho_0 L\dot{c}$. Thus $dT^*$ satisfies

$$\int_{V_{oc}} \rho C_P \frac{\partial T^*}{\partial t}\, dV = -Q^* + 4\pi c^2 \dot{c}\rho_0 L\,. \tag{A4}$$

Since convective heat transport is assumed to vanish in the solid inner core, the integral is restricted to the fluid outer core (though it differs little from an integral over the entire core since $c^3 \ll b^3$). The bulk contribution in (A1) to $\partial\rho/\partial t$ due to thermal convective rearrangement is then given by

$$\frac{\partial \rho}{\partial t} = -\frac{\alpha}{C_P}\left(\rho C_P \frac{\partial T^*}{\partial t}\right)\,, \tag{A5}$$

which can be related to the cooling through (A4). Since the temperature change through the core is nearly constant, we define a mass-averaged convective rate of change of temperature $\overline{dT}^*/dt$ by

$$\frac{\overline{dT}^*}{dt} = M_{oc}^{-1} \int_{V_{oc}} \rho \frac{\partial T^*}{\partial t}\, dV\,, \tag{A6}$$

where $M_{oc}$ is the mass of the outer core, and approximate the bulk contribution to (A1) by

$$\alpha \frac{\overline{dT}^*}{dt} \int_{V_{oc}} \rho\psi\, dV\,. \tag{A7}$$

Combining (A2)–(A7), we may thus write the thermal contribution to the release of gravitational energy as

$$\Phi_T = -\frac{\alpha}{C_P}\left\{Q^*[\overline{\psi} - \psi(b)] - 4\pi c^2 \dot{c}\rho_0 L[\overline{\psi} - \psi(c)]\right\}\,. \tag{A8}$$

The gravitational energy released by chemical fractionation is most naturally related to the change in the inner-core radius. As the inner core solidifies, the mass of the outer core decreases by

$$\frac{dM_{oc}}{dt} = -4\pi c^2 \rho_h \frac{dc}{dt}\,, \tag{A9}$$

where $\rho_h$ is the density of the heavy component at ambient pressure and temperature. The corresponding change in the fluid composition is determined by differentiating (21) to obtain

$$\frac{1}{C}\frac{dC}{dt} = -\frac{1}{M_{oc}}\frac{dM_{oc}}{dt}\,. \tag{A10}$$

The changes in composition cause changes in density, which, for an ideal mixture, can be calculated from

$$C = \left(\frac{1}{\rho_h} - \frac{1}{\rho}\right) \Big/ \left(\frac{1}{\rho_h} - \frac{1}{\rho_l}\right) \tag{A11}$$

(compare (43)–(45)). By differentiating (A11) to show that

$$\frac{1}{C}\frac{dC}{dt} = \frac{\rho_h}{\rho(\rho - \rho_h)}\frac{d\rho}{dt} \tag{A12}$$

and combining this result with (A9) and (A10), we find that the bulk change of density due to composition is given by

$$\frac{\partial \rho}{\partial t} = -\frac{4\pi c^2(\rho_h - \rho)\rho}{M_{oc}}\frac{dc}{dt}\,. \tag{A13}$$

On substituting (A13) into (A1) and including the density change at the ICB on detachment of the compositional boundary layer, we obtain

$$\Phi_C = 4\pi c^2 \Delta\rho[\overline{\psi} - \psi(c)]\frac{dc}{dt} \tag{A14}$$

for the gravitational energy released by compositional rearrangement, where $\Delta\rho$ is the (positive) density jump due solely to changes in composition (and not phase) across the ICB. The total gravitational energy release is $\Phi \equiv \Phi_T + \Phi_C$.

Note that the physical arguments and calculations for the release of gravitational energy by thermal convection are completely analogous to those for compositional convection. The expression for $\Phi_C$ is equivalent to previous expressions for compositional energy release, and we argue here that $\Phi_T$ should also be included. The effects of adiabatic conduction are analogous, though somewhat larger, to those of barodiffusion.

## Appendix B: Redistribution of Entropy

We estimate here the effects of advective redistribution of entropy in (14) and show that they are equal in magnitude to the ohmic heat produced by thermal convection. We also discuss the connection between the approach based on energy which we have used and previous approaches based on entropy.

Since (7) holds for arbitrary volumes $V$, the local entropy equation is given by

$$\frac{Ds}{Dt} = \frac{-\nabla \cdot \mathbf{q} + J^2/\sigma}{\rho T} \equiv E\,, \tag{B1}$$

where $E$ denotes the local rate of entropy production and effects due to compositional diffusion are neglected. Define the mass average of a variable $X$ by

$$\overline{X} = \int_V \rho X\, dV \Big/ \int_V \rho\, dV \tag{B2}$$

so that the integral of (B1) gives $d\overline{s}/dt = \overline{E}$. If the core remains nearly isentropic, then

$$\frac{\partial s}{\partial t} = \frac{d\overline{s}}{dt} = \overline{E}\,. \tag{B3}$$

Hence, by subtraction of (B3) from (B1), we can identify the advective entropy flux required to keep the core well mixed as

$$\mathbf{v}\cdot\nabla s = E - \overline{E}\,. \tag{B4}$$

Thus the advective term in (14) becomes

$$\overline{T\mathbf{v}\cdot\nabla s} = \overline{TE} - \overline{T}\,\overline{E} = \overline{(T-\overline{T})(E-\overline{E})}\,, \tag{B5}$$

which contributes to the heat balance if there are persistent correlated spatial variations in the rate of entropy production and variations in temperature across the core.

The temperature variation in the core from the mass average is small (about 10% of $\overline{T}$ at the CMB and 16% at the Earth's center). However, the largest change in entropy is due to cooling, and if $Nu \neq 1$, then a significant amount of this occurs at the CMB. Consequently, $E-\overline{E}$ is not small, and the advective term $T\mathbf{v}\cdot\nabla s$ contributes significantly to the heat balance. We have argued that this contribution exactly balances the ohmic heat $\Phi_T$ produced by thermal convection, as can now be confirmed to within the approximations of our analytical model.

The entropy production due to cooling consists of three parts. The first two represent the entropy production that occurs in the thin thermal boundary layer at $r = b$, through which the convective heat flux $\mathbf{q}^*$ is transferred into the mantle, and that which occurs due to latent-heat release at $r = c$ on solidification of the inner core. The remaining part occurs uniformly throughout the core and is due both to the divergence of the conductive heat flux $\mathbf{q} - \mathbf{q}^*$ along the adiabatic gradient and to the final small-scale diffusion of well-mixed thermal and compositional anomalies. The entropy production due to ohmic dissipation is smaller than these terms by a factor $O(\phi)$ and its correlation (if any) with $T - \overline{T}$ can be neglected. Thus we write

$$E = \frac{-q^*\delta(r-b)}{\rho_0 T(b)} + \frac{L\dot{c}\delta(r-c)}{T(c)} + E_a\,, \tag{B6}$$

where the source terms at the boundaries are approximated as delta functions and $E_a$ is a constant that describes the effects of conduction along the adiabat and small-scale diffusion. Hence the average rate of entropy production is given by

$$\overline{E} = \frac{3}{4\pi b^3}\left(\frac{-Q^*}{\rho_0 T(b)} + \frac{4\pi c^2 \dot{c} L}{T(c)}\right) + E_a\,. \tag{B7}$$

From (23) the adiabatic temperature in the core may be approximated by

$$T(r) = T(c)\left\{1 - \phi\left(r^2 - c^2\right)/b^2\right\} + O(\phi^2)\,. \tag{B8}$$

Using the identities $\phi = Ab^2\gamma_0/K_0$ and $\gamma_0/K_0 = \alpha/(\rho_0 C_P)$ and substituting for the gravitational potential $\psi$ from (30), we can rewrite (B8) as

$$\begin{aligned} T(r) &= T(b)\{1 - \alpha[\psi(r) - \psi(b)]/C_P\} + O(\phi^2) \\ &= T(c)\{1 - \alpha[\psi(r) - \psi(c)]/C_P\} + O(\phi^2) \end{aligned} \tag{B9}$$

from which we can calculate $\overline{T}$. We then combine (B5), (B6), (B7) and (B9) to obtain $\overline{T\mathbf{v}\cdot\nabla s}$. It follows that the advective contribution to the heat budget is

$$\int_V \rho T\mathbf{v}\cdot\nabla s\,dV =$$

$$-\frac{\alpha}{C_P}\left\{Q^*\left[\overline{\psi} - \psi(b)\right] - 4\pi c^2\dot{c}\rho_0 L\left[\overline{\psi} - \psi(c)\right]\right\}, \tag{B10}$$

which is equal to the thermal contribution $\Phi_T$ to the ohmic dissipation as given by (11). Thus, to within the approximations of this calculation, these two contributions to the heat budget cancel as expected. As shown in section 2.3, this cancellation is actually exact due to the fact that the change in total energy between a sequence of known states is independent of the internal processes that produce the change.

It may also be noted that a number of authors [e.g., *Backus*, 1975; *Hewitt et al.*, 1975; *Gubbins et al.*, 1979] have estimated the ohmic heating associated with the geodynamo from the global entropy equation rather than from the global energy equation, which we have used here. A simple expression of the global entropy equation is obtained by integrating (B1) over the core to obtain

$$\int_V \rho\frac{Ds}{Dt}\,dV = -\frac{1}{T(b)}\int_S \mathbf{q}\cdot d\mathbf{S} + \int_V \frac{J^2}{\sigma T}\,dV + \int_V k\left(\frac{\nabla T}{T}\right)^2 dV\,, \tag{B11}$$

where $k$ is the thermal conductivity. Thus the heat flux across the CMB is a sink of entropy for the core, while ohmic heating and conduction down temperature gradients are entropy sources. (The latter must include both conduction down the adiabat and the significant contribution from small-scale conduction from convecting thermal anomalies.) Since (7) and (B11) are both derived by integration of (B1), the two methods of estimating the ohmic heating are essentially equivalent. Though the energy method leads to an estimate of $\overline{J^2/\sigma}$ and the entropy method to an estimate of $\overline{T}\;\overline{J^2/\sigma T}$, the two estimates differ by no more than 10% due to the relative uniformity of temperature. Calculations based solely on global entropy and global energy are equally incapable of distinguishing how much of the evolution of the core through nearly well-mixed, adiabatic, hydrostatic states can be attributed to internal diffusive and convective processes. In each case the global equation reflects only the change in the mean state of the core and information about convective processes must come from the internal mechanics.

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