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On Howard's technique for perturbing neutral solutions of the Taylor–Goldstein equation

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Howard's technique for examining the stability characteristics of a two-dimensional, inviscid, heterogeneous shear flow in the vicinity of a neutral curve is considered. Examples are presented for which erroneous conclusions would be obtained by a direct interpretation of the results of this technique. Examples for which *instability* would be deduced for a stable region; an example for which *stability* would be deduced for an unstable region; and an example for which the technique breaks down altogether are discussed.

The last example is plane Couette flow between two rigid walls, which is shown to be destabilized by the addition of stable stratification. This example thus proves that the existence of a relative extremum in the basic vorticity profile is *not* a necessary condition for the instability of an inviscid, heterogeneous shear flow.

1. Introduction

The theoretical investigation of the stability of a parallel, two-dimensional, inviscid, heterogeneous shear flow to infinitesimal disturbances requires the solution of the Taylor–Goldstein equation

$$\phi'' + \left[\frac{JN^2(y)}{(U-c)^2} - \frac{U''}{U-c} - \alpha^2\right]\phi = 0. \tag{1.1}$$

Here the disturbance is considered to have a stream function $\phi(y)\,e^{i\alpha(x-ct)}$ in a basic flow $U(y)$ with buoyancy frequency $N(y)$ and (representative) Richardson number J. All variables in (1.1) have been non-dimensionalized with respect to an intrinsic length scale L and velocity scale V, and the Boussinesq approximation has been made. The specification of zero normal velocity at the boundaries, $y = y_1$ and y_2 say, requires further

$$\phi = 0 \quad (y = y_1, y_2). \tag{1.2a, b}$$

Equations (1.1) and (1.2) constitute an eigenvalue problem, which is usually stated in the following form. Given real, non-negative values of α and J, determine the eigensolutions $\phi(y)$ and $c \equiv c_r + ic_i$, if such solutions exist. Curves in the α, J plane along which $c_i = 0$ are called neutral curves, and for any point on these the disturbance neither grows nor decays. For each value of α and J, if there exists a solution (ϕ, c) there is a corresponding solution (ϕ^*, c^*). However, only the solutions with $c_i \geqslant 0$ are appropriate, because only these solutions represent the limit of solutions of the viscous equations as the viscosity tends to zero. Alternatively, it is only solutions with $c_i \geqslant 0$ which are approached by solutions of an inviscid initial-value problem. Thus, a point (α, J) is unstable if there exists a solution with $c_i > 0$ and stable if either there exists a solution with $c_i = 0$ or there is no solution at all.

A large majority of the analytically known solutions of the Taylor–Goldstein equation are valid only on neutral curves. This is because frequently along such a curve c is constant, often zero. The determination of whether a particular neutral curve has contiguous unstable modes or is an isolated neutral curve requires additional investigation. Miles (1963) carried out such an investigation by examining the solution just off the neutral curve. This required a fairly detailed perturbation expansion of (1.1) by a method not applicable to all profiles of $U(y)$ and $N(y)$.† However, stimulated by this work, Howard (1963) developed simple, general formulae for calculating the values of $\partial c/\partial \alpha$ and $\partial c/\partial J$ along a neutral curve. By examining the sign of the imaginary parts of these expressions, one was to determine the characteristics of the solution on either side of the neutral curve. These formulae, once understood, are very easy to use and have assisted many an investigation.

The purpose of this note is to present examples for which the straightforward interpretation of the results of applying Howard's technique leads to erroneous conclusions. We present examples for which *instability* would be deduced for a stable region and, conversely, an example for which *stability* would be deduced for an unstable region. We also present an example for which the procedure breaks down completely. The explanation for each of these occurrences is different and is discussed in detail.

Although we present only four examples in detail, it is easy to construct many other examples for which application of Howard's technique leads to erroneous conclusions of one of the above types.

Before examining the particular examples, let us consider the essentials of Howard's technique. Multiplying (1.1) by ϕ, integrating the result between y_1 and y_2 and rearranging, we obtain

$$I(c, \phi) = \frac{\int_{y_1}^{y_2} \{[JN^2(U-c)^{-2} - U''(U-c)^{-1}]\,\phi^2 - \phi'^2\}\,dy}{\int_{y_1}^{y_2} \phi^2\,dy} = \alpha^2. \quad (1.3a, b)$$

Now, for *fixed* α, J and c, I is stationary with respect to first-order variations in ϕ about a solution of (1.1), provided that the variations vanish at $y = y_1, y_2$. Thus, using the chain rule to differentiate (1.3b) with respect to α, we obtain

$$\frac{\partial c(\alpha, J)}{\partial \alpha} = \frac{2\alpha \int_{y_1}^{y_2} \phi^2\,dy}{\int_{y_1}^{y_2} [2JN^2(U-c)^{-3} - U''(U-c)^{-2}]\,\phi^2\,dy}. \quad (1.4)$$

† It was essential, for example, for $U(y)$ to be monotonic.

Using a subscript s to denote evaluation on a neutral curve, we could, in principle, take the limit $\alpha \to \alpha_s$, $c \to c_s + i0+$ in (1.4) to obtain a formula for $\partial c/\partial \alpha$ along the curve. In practice, however, the singularities which are generally present in ϕ (Miles 1961) would make evaluation of the integrals awkward. This is avoided by introducing the smooth function H, related to ϕ by $H = (U-c)^{n-1}\phi$, where n is *one* of the values satisfying $n(1-n)\,U'^2 = JN^2$ at $y = y_c$ with $U(y_c) = C$. (Only one of the two solutions will make H smooth.) Equation (1.4) then becomes

$$\left(\frac{\partial c}{\partial \alpha}\right)_s = \lim_{c\to c_s+i0+} \frac{2\alpha \displaystyle\int_{y_1}^{y_2} H^2(U-c)^{2-2n}\,dy}{\displaystyle\int_{y_1}^{y_2} [2JN^2(U-c)^{-1} - U'']\,(U-c)^{-2n} H^2\,dy}. \tag{1.5}$$

If the neutral curve is one along which c is a constant, then it is described by $J = J_s(\alpha)$, and appropriate differentiation leads to

$$\left(\frac{\partial c}{\partial J}\right)_s = -\frac{1}{J_s'}\left(\frac{\partial c}{\partial \alpha}\right)_s. \tag{1.6}$$

Equations (1.5) and (1.6) are the two Howard formulae.

2. The specific examples

2.1. *Examples for which the technique implies instability of a stable region*

The profiles $$U = \sin y, \quad N^2 = 1, \tag{2.1a,b}$$

with $$-y_1 = y_2 = \pi \tag{2.2a,b}$$

upon substitution in (1.1) and (1.2) yield the neutral eigensolution

$$c = 0, \quad \phi = (\sin y)^{(1-\alpha^2)^{\frac{1}{2}}}, \tag{2.3a,b}$$

$$J \equiv J_1(\alpha) = (1-\alpha^2)^{\frac{1}{2}} - 1 + \alpha^2 \quad (0 \leqslant \alpha \leqslant 1). \tag{2.3c}$$

Applying Howard's technique, with $n = 1-(1-\alpha^2)^{\frac{1}{2}}$ and $H = 1$, we obtain

$$\frac{\partial c}{\partial \alpha} = -\frac{2i\alpha \cot(\pi\beta)\,B(\beta+\frac{1}{2},\frac{1}{2})}{(1-\beta)(1+2\beta)\,B(\beta,\frac{1}{2})} \tag{2.4a}$$

and $$\frac{\partial c}{\partial J} = -\frac{2i\beta \cot(\pi\beta)\,B(\beta+\frac{1}{2},\frac{1}{2})}{(1-\beta)(1-4\beta^2)\,B(\beta,\frac{1}{2})}, \tag{2.4b}$$

where $$\beta = (1-\alpha^2)^{\frac{1}{2}}. \tag{2.5}$$

The interpretation of (2.4) seems clear: for values of J slightly less than $J_1(\alpha)$ the Taylor–Goldstein equation has an unstable solution. Numerical investigation (Hazel 1972) shows this to be false. There is an unstable mode contiguous to $J = J_1(\alpha)$ only for $0 < \alpha < \frac{1}{2}\sqrt{3}$; the portion $\frac{1}{2}\sqrt{3} < \alpha < 1$ is an isolated neutral curve. There is another neutral eigensolution (Thorpe 1969)

$$c = 0, \quad \phi = (\cos \tfrac{1}{2}y)^{\frac{1}{2}\pm(\frac{1}{4}-J)^{\frac{1}{2}}}(\sin \tfrac{1}{2}y)^{\frac{1}{2}\mp(\frac{1}{4}-J)^{\frac{1}{2}}}, \quad \alpha = \tfrac{1}{2}\sqrt{3} \quad (0 \leqslant J \leqslant \tfrac{1}{4}) \tag{2.6a,b,c}$$

and Hazel shows that only the region

$$0 < \alpha < \tfrac{1}{2}\sqrt{3}, \quad 0 \leqslant J < J_1(\alpha)$$

bounded by the two neutral curves is unstable. Howard's technique applied along $\alpha = \frac{1}{2}\sqrt{3}$ with $n = \frac{1}{2} \pm (\frac{1}{4} - J)^{\frac{1}{2}}$ and $H = (\cos\frac{1}{2}y)^{\pm 2(\frac{1}{4}-J)^{\frac{1}{2}}}$ yields

$$\partial c/\partial\alpha = \mp 2 \times 3^{\frac{1}{2}}\, i(\tfrac{1}{4} - J)^{\frac{1}{2}}. \tag{2.7}$$

The upper sign implies instability for $\alpha < \frac{1}{2}\sqrt{3}$, in agreement with the numerical results. The lower sign would seem to imply that contiguous to the neutral eigensolution

$$c = 0, \quad \phi = (\cos\tfrac{1}{2}y)^{\frac{1}{2}-(\frac{1}{4}-J)^{\frac{1}{2}}}(\sin\tfrac{1}{2}y)^{\frac{1}{2}+(\frac{1}{4}-J)^{\frac{1}{2}}}, \quad \alpha = \tfrac{1}{2}\sqrt{3} \quad (0 \leqslant J < \tfrac{1}{4}) \quad (2.6'a,b,c)$$

there is an unstable mode for $\alpha > \frac{1}{2}\sqrt{3}$. This is false. The solution (2.6′) represents an isolated neutral eigensolution, and we note that the eigenfunction does *not* approach

$$\phi = \cos\tfrac{1}{2}y \quad \text{as} \quad J \to 0, \tag{2.8}$$

which is the neutral eigenfunction in the homogeneous ($J \equiv 0$) situation.

An exactly analogous situation occurs for the profiles

$$U = \sin y, \quad N^2 = \cos^2 y, \tag{2.9a,b}$$

with

$$-y_1 = y_2 = \pi. \tag{2.10}$$

There is a neutral eigensolution (Thorpe 1969)

$$c = 0, \quad \phi = (\sin y)^{1-\alpha^2}, \quad J \equiv J_2(\alpha) = \alpha^2(1-\alpha^2) \quad (0 \leqslant \alpha \leqslant 1), \quad (2.11a,b,c)$$

for which Howard's technique, on taking $n = \alpha^2$, $H = 1$, yields

$$\frac{\partial c}{\partial\alpha} = \frac{2i\cot(\pi\alpha^2)\,B(\frac{3}{2}-\alpha^2,\frac{1}{2})}{\alpha B(1-\alpha^2,\frac{1}{2})} \tag{2.12a}$$

and

$$\frac{\partial c}{\partial J} = \frac{-i\cot(\pi\alpha^2)\,B(\frac{3}{2}-\alpha^2,\frac{1}{2})}{\alpha^2(1-2\alpha^2)\,B(1-\alpha^2,\frac{1}{2})}. \tag{2.12b}$$

Not all values of J slightly less than $J_2(\alpha)$ are unstable, as would be inferred from (2.12), only those for which $0 < \alpha < \frac{1}{2}\sqrt{2}$. There is another neutral eigensolution

$$c = 0, \quad \phi = (\cos\tfrac{1}{2}y)^{\frac{1}{2}\pm(\frac{1}{4}-J)^{\frac{1}{2}}}(\sin\tfrac{1}{2}y)^{\frac{1}{2}\mp(\frac{1}{4}-J)^{\frac{1}{2}}}, \tag{2.13a,b}$$

$$J \equiv J_3(\alpha) = \tfrac{3}{4} - \alpha^2 \quad (\tfrac{1}{2}\sqrt{2} \leqslant \alpha \leqslant \tfrac{1}{2}\sqrt{3}) \tag{2.13c}$$

and only the region

$$0 < J < \min[J_2(\alpha), J_3(\alpha)] \quad (0 < \alpha < \tfrac{1}{2}\sqrt{3})$$

is unstable (Hazel 1972). And, as before, it is only the upper-signed solution of (2.13) which is the limit of contiguous unstable eigensolutions.

At first sight, these examples might seem to contradict Miles's (1963) theorem (viii): "the existence of a neutral curve in an (α, J)-plane implies the existence of contiguous, complex eigenvalues" with the added footnote: "Theorem (viii) does not guarantee the existence of unstable eigenvalues, although their existence is implied by the development...below", i.e. the complex eigenvalue c of the theorem has non-zero imaginary part. However, this theorem was proved only for monotonic velocity profiles, essentially because the proof required the value of c along the neutral curve to be different from the value that $U(y)$ takes at either of the boundaries. If the velocity profile is not monotonic, then, as numerical investigation of the above examples show, the existence of a neutral curve does not necessarily imply the existence of contiguous eigensolutions. Now, if there are no contiguous eigensolutions, then the manipulations involved in arriving at the Howard formulae are meaningless; there are just no solutions adjacent to the neutral curve for which first variations can be considered, to yield (1.4), and $\partial c/\partial\alpha$ and $\partial c/\partial J$ have no significance. In such situations, the right-hand sides of (1.5) and (1.6) [(2.4), the lower-signed solution of (2.7) and (2.12) in our examples] merely represent numbers obtained by a formal evaluation of integrals having no foundation.

2.2. *An example for which the technique implies stability of an unstable region*

Thorpe (1969) in a footnote comments upon the profiles

$$U = \operatorname{sech}^2 y, \quad N^2 = \operatorname{sech}^4 y \tag{2.14a, b}$$

with

$$-y_1 = y_2 = \infty. \tag{2.15}$$

He remarks that "[there exist] the (degenerate) neutral eigensolutions

$$[c = 0], \quad \phi = \tanh y \operatorname{sech} y, \quad J = 3+\alpha^2 \tag{2.16a, b, c}$$

and

$$[c = 0], \quad \phi = \operatorname{sech}^2 y, \quad J = \alpha^2. \tag{2.17a, b, c}$$

These do not appear to be stability boundaries". Indeed not. Applying Howard's technique to the neutral solutions (2.16) and (2.17), with $n = 1$ and $H = \phi$, we obtain

$$\frac{\partial c}{\partial \alpha} = \frac{\partial c}{\partial J} = 0 \quad (J = 3+\alpha^2) \tag{2.18a, b}$$

and

$$\frac{\partial c}{\partial \alpha} = \frac{2\alpha}{3J}, \quad \frac{\partial c}{\partial J} = -\frac{1}{3J} \quad (J = \alpha^2). \tag{2.19a, b}$$

Equations (2.16) and (2.17) would seem to indicate that c does not become complex just off the neutral curves. Further, $c = 0$, the value of $U(\pm\infty)$, and the examples of § 2.1 might lead one to conclude that both (2.16) and (2.17) represent isolated neutral curves.

However, there are two further neutral eigensolutions:

$$c = 2(\alpha^2+1)(\alpha^2+3\alpha+2)^{-1}, \quad \phi = (\operatorname{sech}^2 y - c)^{\frac{1}{2}(1-\alpha)} \operatorname{sech}^\alpha y \tanh y, \tag{2.20a, b}$$

$$J = \alpha(\alpha^2-4\alpha+3)(\alpha+2)^{-1} \quad (0 \leqslant \alpha \leqslant 1) \tag{2.20c}$$

and

$$c = 2(\alpha-1)(\alpha+1)^{-1}, \quad \phi = (\operatorname{sech}^2 y - c)^{1-\frac{1}{2}\alpha} \operatorname{sech}^\alpha y, \tag{2.21a, b}$$

$$J = \alpha(\alpha^2-5\alpha+6)(\alpha+1)^{-1} \quad (1 \leqslant \alpha \leqslant 2), \tag{2.21c}$$

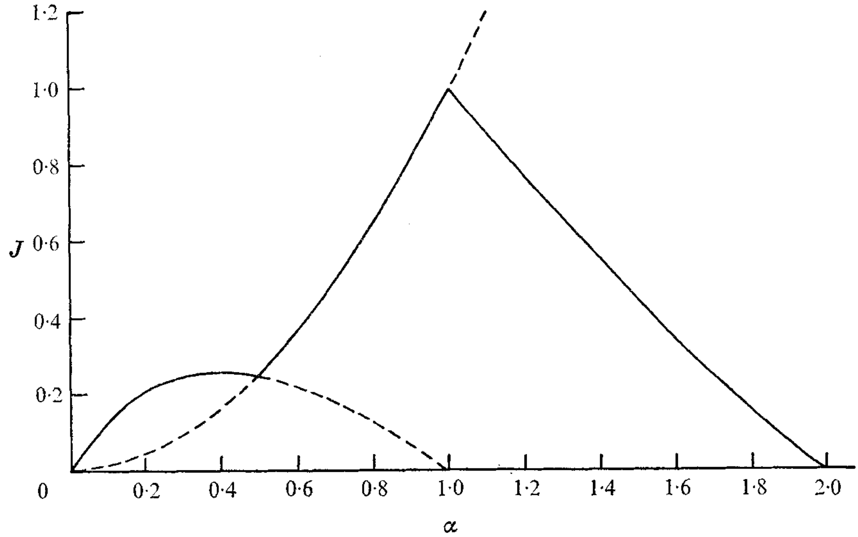

FIGURE 1. The stability boundary (solid line) and some neutral curves (broken or solid lines) for $U(y) = \operatorname{sech}^2 y$, $N^2(y) = \operatorname{sech}^4 y$, $-\infty < y < \infty$.

as can be verified by direct substitution. Numerical investigation shows that the stability boundary (the curve enclosing the region of instability) is made up of: (2.21); part of (2.20); and part of (2.17) [and none of (2.16)]. This is depicted in figure 1, from which it is clear that the neutral curve (2.17*c*) has contiguous unstable modes only along that part for which $0 < \alpha < 1$.

2.3. *An example for which the technique breaks down completely*

We discuss here an example for which the evaluation of the integrals in (1.5) leads to $\partial c/\partial \alpha = \infty$.

Consider the profiles

$$U = y, \quad N^2 = y^2 \tag{2.22a, b}$$

with

$$-y_1 = y_2 = \pi, \tag{2.23}$$

which lead to the eigenvalue problem†

$$\phi'' + [Jy^2(y-c)^{-2} - \alpha^2]\phi = 0, \tag{2.24}$$

$$\phi = 0 \quad (y = \pm\pi). \tag{2.25a, b}$$

There are two infinite families of neutral eigensolutions given by

$$c = 0, \quad \phi = \cos(n-\tfrac{1}{2})y, \quad J = (n-\tfrac{1}{2})^2 + \alpha^2, \tag{2.26a, b, c}$$

and

$$c = 0, \quad \phi = \sin my, \quad J = m^2 + \alpha^2, \tag{2.27a, b, c}$$

where

$$m, n = 1, 2, \ldots. \tag{2.28}$$

† The profiles have been considered previously, in a different context, by Høiland & Riis (1968). They constrained attention to small values of c and thus the culmination of our stability calculation, figure 2, does not appear in their work.

Applying Howard's technique, with $n = 1$ and $H = \phi$, we obtain

$$(\partial c/\partial \alpha)_{J=(n-\frac{1}{2})^2+\alpha^2} = -i\alpha J^{-1} \tag{2.29}$$

and
$$(\partial c/\partial \alpha)_{J=m^2+\alpha^2} = \infty. \tag{2.30}$$

Equation (2.30) is obviously unsatisfactory, reflecting the fact, as will be shown below, that $c \to 0$ as $\alpha \to \alpha_m \equiv (J-m^2)^{\frac{1}{2}}$ less rapidly than $\alpha - \alpha_m$.

The explicit behaviour of the eigenvalue c in the neighbourhood of the neutral curve (2.27*c*) can be obtained by expressing the solution of (2.24) in terms of two confluent hypergeometric functions:

$$\phi = e^{-\frac{1}{2}ry}(y-c)^{\frac{1}{2}q}\{A\,\Phi[p,q,r(y-c)] + B\Psi[p,q,r(y-c)]\}, \tag{2.31}$$

where
$$p = \tfrac{1}{2} + (\tfrac{1}{4} - Jc^2)^{\frac{1}{2}} - cJ(\alpha^2 - J)^{-\frac{1}{2}}, \tag{2.32a}$$

$$q = 1 + (1 - 4Jc^2)^{\frac{1}{2}}, \tag{2.32b}$$

and
$$r = 2(\alpha^2 - J)^{\frac{1}{2}}. \tag{2.32c}$$

The eigenvalue relationship is thus

$$\Phi[r(\pi - c)]\,\Psi[-r(\pi + c)] = \Phi[-r(\pi + c)]\,\Psi[r(\pi - c)], \tag{2.33}$$

where the two arguments p and q have been suppressed. Expanding (2.33) about (2.26*c*) and using the definition of $\alpha_n \equiv [J - (n-\frac{1}{2})^2]^{\frac{1}{2}}$, we obtain

$$c \sim -\tfrac{1}{2}iJ^{-1}(\alpha^2 - \alpha_n^2) \quad (\alpha \to \alpha_n) \tag{2.34}$$

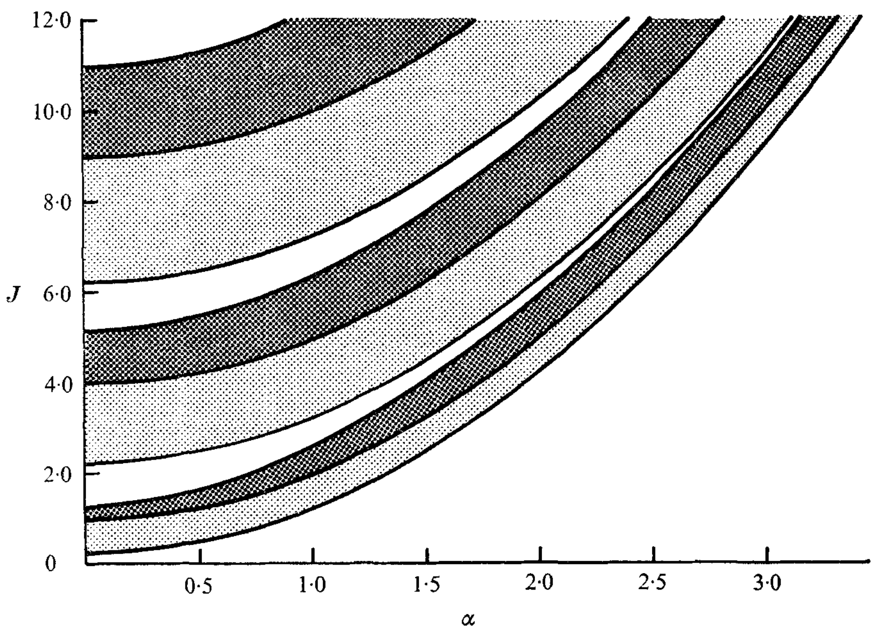

FIGURE 2. The stability characteristics for $U(y) = y$, $N^2(y) = y^2$, $-\pi \leqslant y \leqslant \pi$. The unshaded regions are stable, the lightly shaded regions are unstable with $c_r = 0$ and the darkly shaded regions are unstable with $c_r \neq 0$.

in agreement with (2.29), while expanding about (2.27c), we obtain†

$$c \sim \left[\frac{6Jm}{\pi}\operatorname{Si}(2m\pi) - J^2\int_0^1 \cos 2m\pi t \log^2\left(\frac{t}{1-t}\right)dt - \frac{2J^2}{m\pi}\operatorname{Cin}(2m\pi)\operatorname{Si}(2m\pi)\right]^{-\frac{1}{2}} (\alpha^2-\alpha_m^2)^{\frac{1}{2}} \quad (\alpha\to\alpha_m), \quad (2.35)$$

where the cosine and sine integrals are respectively defined by

$$\operatorname{Cin}(x) = \int_0^x \frac{1-\cos t}{t}\,dt, \quad \operatorname{Si}(x) = \int_0^x \frac{\sin t}{t}\,dt. \quad (2.36a, b)$$

Numerical solution of (2.33) yields figure 2.

This flow is an interesting example of an inviscid velocity profile which is destabilized by the addition of a stable stratification, being unstable only if the Richardson number is *larger* than $\frac{1}{4}$. It also represents a counter example to the oft-made conjectural extension of Rayleigh's flex-point theorem, that, even for a heterogeneous shear flow, a necessary condition for instability is that $U''(y)$ must change sign somewhere in the flow field. This is now clearly not correct.

3. Conclusion

We conclude that, although Howard's technique for investigating the stability characteristics in the neighbourhood of a neutral curve is often very effective, it should be used with caution. Examples exist for which erroneous conclusions would be obtained by a direct interpretation of the results of the technique.

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† Banks & Drazin in a forthcoming paper obtain a relationship equivalent to (2.35) by an alternative, more general, method.