articles

Transitions in double-diffusive convection

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The transitions of solutions of differential equations from one stability regime to another are of great current interest, both mathematical and physical, and there are conflicting hypotheses as to how such transitions occur. Here we present the results of an investigation of double-diffusive convection, which is important in oceanography, astrophysics and chemical engineering. The calculated transitions are found to be very different from those previously suggested.

MOST fluid motion is turbulent. One approach that has been used in investigating this form of motion is to examine the transitions undergone by an initially laminar flow in its evolution to a turbulent state. Landau[1] hypothesised that as some appropriate non-dimensional parameter, R, increases, a critical value is reached at which the particular time-independent laminar flow under investigation becomes unstable and equilibrates to a new time-dependent flow. At a larger critical value this flow itself becomes unstable and this process of transitions is envisaged to continue until the flow has become so complicated that it would be referred to as turbulent. An alternative hypothesis has been recently suggested by Ruelle and Takens[2], whereby after a relatively small number of transitions, a well defined value of R is reached at which the flow exhibits an abrupt transition to a far more complicated, random, and hence turbulent, motion. Since these are both abstract hypotheses, only a direct investigation of the governing equations of motion can decide if either of these descriptions is correct, or if a different set of transitions takes place. In an investigation of this sort, but motivated by biological considerations, May[3] recently determined the transitions that occur in a number of first-order nonlinear difference equations. He found that as the appropriate R increases through a sequence of critical values, the solution changes from consisting of one stable equilibrium point, through stable cycles of period 2^n ($n=1,2,\ldots$), and then, at a finite value of R, there is chaos, with slightly different initial conditions leading to solutions which diverge with time. We present here the results of explicit calculations of the transitions that occur in two-dimensional double-diffusive convection. The motion is governed by a set of coupled nonlinear partial differential equations and the transitions that occur are different in form from the three discussed above.

Double-diffusive convection

Double-diffusive convection is a generic term for the type of convection that occurs in fluids in which there are two components of different molecular diffusivities which contribute in an opposing sense to the vertical density gradient. For different sets of components, this form of convection has an important role in oceanography, astrophysics and chemical engineering[4]. Here we use the terminology of heat and salt, the components appropriate to oceanography. We restrict attention to the Rayleigh–Bénard problem[4], where the fluid is considered to occupy the space between two infinite planes separated by a distance D, with the upper plane maintained at temperature T_0 and salinity S_0 and the lower plane maintained at temperature $T_0+\Delta T$ and salinity $S_0+\Delta S$ (ΔT, $\Delta S > 0$). We assume both planes to be stress free and perfect conductors of heat and salt, and restrict attention to two-dimensional motion, dependent only on one horizontal coordinate and the vertical coordinate. Expressing all lengths in units of D, time in units of D^2/κ_T (where κ_T is the thermal diffusivity) and representing the temperature T^* and salinity S^* by

$$T^* = T_0+\Delta T(1-z+T); \quad S^* = S_0+\Delta S(1-z+S)$$

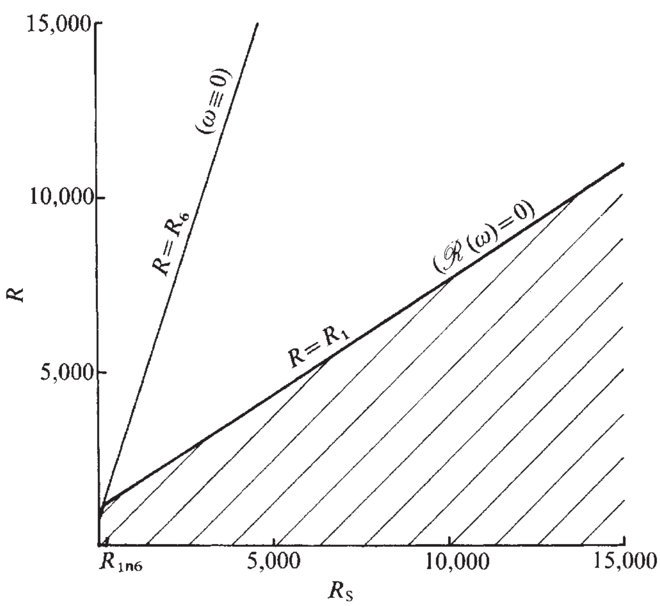
we can write the governing Boussinesq equations of motion in terms of a stream function ψ as

$$\begin{aligned} \sigma^{-1}\nabla^2\partial_t\psi-\sigma^{-1}J(\psi,\nabla^2\psi) &= -R\partial_xT+R_S\partial_xS+\nabla^4\psi \\ \partial_tT+\partial_x\psi-J(\psi,T) &= \nabla^2T \\ \partial_tS+\partial_x\psi-J(\psi,S) &= \tau\nabla^2S \\ \psi = \partial_{zz}{}^2\psi = T = S &= 0 \ (z=(0,1)) \end{aligned}$$

where

$$J(f,g) = \partial_xf\partial_zg-\partial_zf\partial_xg$$

Fig. 1 The results of linear stability analysis for $\sigma = 1$ and $\tau = 10^{-\frac{1}{2}}$. Overstability first occurs along $R = R_1$ and monotonic stability first occurs along $R = R_6$. Only the hatched region is stable to linear disturbances.



The four non-dimensional parameters appearing in these equations are: the Prandtl number $\sigma = \nu/\kappa_T$, where ν is the kinematic viscosity; the ratio of the diffusivities $\tau = \kappa_S/\kappa_T$, where κ_S, the saline diffusivity, is less than κ_T; the thermal Rayleigh number $R = \alpha g \Delta T D^3/\kappa_T \nu$, where is α the coefficient of thermal expansion, and g is the gravitational acceleration; and the saline Rayleigh number $R_S = \beta g \Delta S D^3/\kappa_T \nu$ where β is the saline analogue of α.

Stability

The criteria for the linear instability of these equations are obtained by neglecting the nonlinear Jacobian terms and representing the solutions in terms of the lowest normal modes with an exponential time dependence of the form $\exp(\omega t)$. The onset of overstability, defined by $\mathscr{R}(\omega) = 0$, first occurs for a value of R which we denote by R_1 and exchange of stabilities, defined by $\omega \equiv 0$, first occurs for a value of R which we denote for convenience by R_6. In both cases the horizontal wavelength of the instability is $2^{3/2}$. In the R–R_S plane the linear stability boundary is a combination of $R = R_1$ and $R = R_6$, as depicted in Fig. 1, which presents a summary of the linearised results for $\sigma = 1$ and $\tau = 10^{-1/2}$. For $R_S > R_{1n6}$, where R_{1n6} is the value of R_S at which $R_1 = R_6$, as R exceeds R_1 the conduction state ($\psi = T = S = 0$) becomes unstable to an oscillatory mode. For $0 < R < R_{1n6}$ as R exceeds R_6 the conduction state becomes unstable to a monotonic mode.

We present here the form of the solutions of horizontal wavelength $2^{3/2}$ which emanate from the above linear transitions, or bifurcation points, as the degree of nonlinearity increases. An appropriate way to describe the resulting solutions is as a function of their amplitude. This is conveniently represented by either the thermal or saline Nusselt numbers evaluated by the lower boundary

$$N_T = 1 - \overline{\partial_z T}|_{z=0} \quad \text{and} \quad N_S = 1 - \overline{\partial_z S}|_{z=0}$$

where the overbar denotes a horizontal average, or by their respective temporal maxima, M_T and M_S. Since for most physical systems $R_S > R_{1n6}$ and this is conceptually the most interesting case, we concentrate on the latter regime and, principally by numerical integration of equations (1), map out the forms of equilibrium solutions in an R–M_S plane in the manner depicted in Fig. 2. The discussion is in general terms; specific details and physical interpretations of the solutions will be published elsewhere (H. E. Huppert and D. R. Moore, in preparation).

The bifurcation point at R_1 can be either supercritical (R increases as M_T or M_S increases) or subcritical (R decreases as M_T or M_S increases). By the straightforward use of modified perturbation theory[5], a very lengthy relationship can be determined (H. E. Huppert and D. R. Moore, in preparation) which indicates, for fixed σ, τ and R_S, which of the two possibilities occurs. It is known from general theory[5] that solutions on branches emanating from subcritical bifurcations are unstable until the branch reaches a minimum value of R. Thereafter the branch continues with the amplitude of the associated time-dependent solutions increasing with increasing R. Supercritical branches are known to support stable solutions. A typical plot of N_T and N_S against time for a solution on a stable portion of the branch sufficiently close to R_1 is shown in Fig. 3*a*.

Increasing *R*

As R increases, this form of motion continues until R reaches a specific value, R_2 say. At $R = R_2$ the solution changes in form and develops a further structure as is indicated in the form of N_T or N_S as a function of time, as plotted in Fig. 3*b*. In both N_T and N_S there are four extrema, two maxima and two minima, per period, where the period is defined in the usual sense as the time between two identical states. In modern mathematical jargon, the solution for $R < R_2$ is on a sphere, while the solution

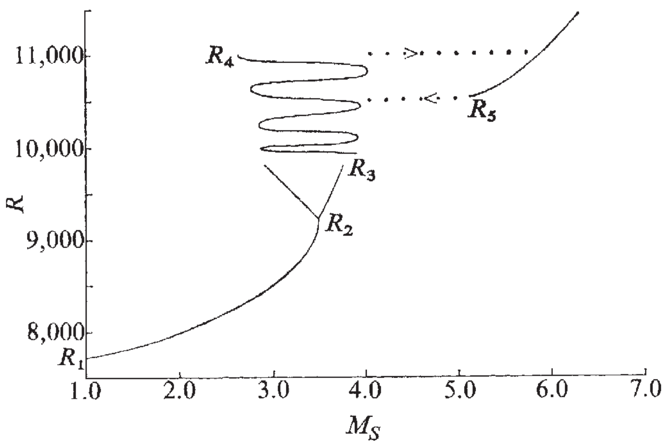
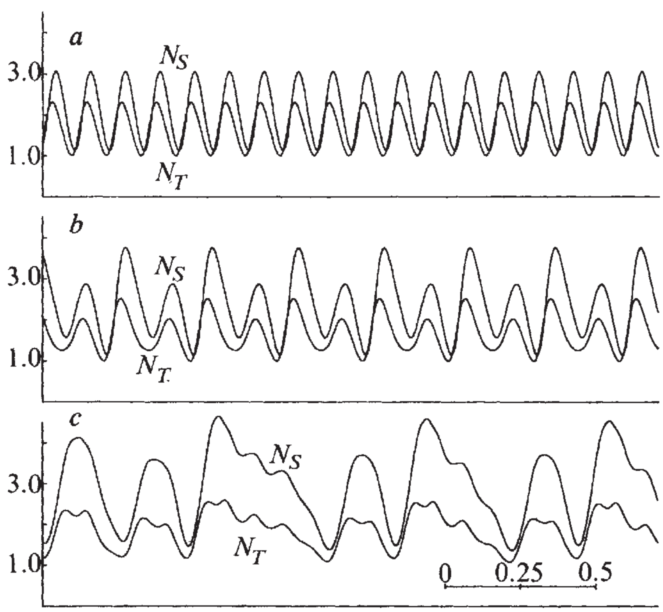


Fig. 2 The stable oscillatory branch and the stable monotonic branch for $\sigma = 1$, $\tau = 10^{-\frac{1}{2}}$ and $R_S = 10^4$. For $R_2 < R < R_3$ the branch is drawn so as to show both local maxima. For $R_3 < R < R_4$ the wavy line indicates that the solution is non-periodic and no definite maximum can be assigned. The dots at $R = R_4$ and $R = R_5$ indicate the jumps in M_S which occur as the solution changes from the oscillatory branch to the monotonic branch.

for $R > R_2$ is on a torus, and the transition at $R = R_2$ is called a bifurcating torus[6]. This form of motion continues until $R = R_3$ say, at which value a transition to a disordered solution occurs. A typical plot of N_T and N_S against time for such a disordered solution is shown in Fig. 3*c*. Long computation runs have not revealed any discernable periodic structure in the solution. Such non-periodic solutions continue to exist for increasing R until, for $R = R_4$ say, the only equilibrium solutions are time independent. For some values of σ, τ and R_S the transition at R_4 occurs before the one at either R_2 or R_3.

For all $R > R_4$ all equilibrium solutions are independent of time. Time-independent motions are more efficient at transporting heat and salt than time-dependent motions and thus the Nusselt numbers undergo a discontinuous increase as the solution changes from being on the oscillatory branch to the monotonic branch, as indicated in Fig. 2. As R increases

Fig. 3 The thermal and saline Nusselt numbers as a function of time for $\sigma = 1$, $\tau = 10^{-\frac{1}{2}}$ and $R_S = 10^4$. *a*, $R_1 < R = 8{,}600 < R_2$; *b*, $R_2 < R = 9{,}800 < R_3$; and *c*, $R_3 < R = 11{,}000$. The horizontal line in the bottom right-hand portion of *c*, represents non-dimensional time.



beyond R_4 the effects of the salt field decrease, and for $R \gg R_4$ the equilibrium solutions approach those for $R_S = 0$, a situation which has been intensively investigated by Moore and Weiss[7].

Decreasing *R*

If R is gradually decreased from some value greater than R_4, the equilibrium monotonic solutions retrace the states that would have been obtained on increasing R from R_4; for each $R > R_4$ there is a unique stable equilibrium solution. If R is decreased below R_4, an equilibrium time-independent solution continues to exist, with decreasing amplitude, until $R = R_5$ say. Further decrease of R leads to a solution on the oscillatory branch already described, or if $R_5 < R_1$ to the null solution. Thus, as indicated in Fig. 2 there is hysteresis between the two different forms of solution.

The time-independent branch of solutions emanates from the bifurcation point at $R = R_6$, which modified perturbation theory shows to be subcritical if $R_S > R_{1n6}$. As mentioned previously, solutions on such a subcritical branch are unstable until the minimum value of R is attained. Thereafter the branch continues and solutions on it are stable, with the amplitude of the solution increasing with increasing R.

If $R_S < R_{1n6}$ only time-independent equilibrium solutions are possible. For sufficiently small R_S the bifurcation at R_6 is supercritical, otherwise it is subcritical (H. E. Huppert and D. R. Moore, in preparation).

Conclusion

To summarise, in the most general case, as R increases there is a transition from the conduction state to an oscillatory motion (Fig. 3*a*), followed by a transition to a more complicated form of oscillatory motion (Fig. 3*b*), followed by a transition to a non-periodic, random state (Fig. 3*c*), followed by a transition to steady motion. Hence by increasing R it is possible in this situation to suppress disordered motions. For some values of σ, τ and R_S a sufficiently large disturbance can cause a transition directly from the conduction state to the steady state, while for other values of σ, τ and R_S only some of the intermediate transitions are omitted.

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Stratified waters as a key to the past

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Density stratification in lakes and oceans generate anoxic conditions below the pycnocline, and sediment facies mirror this development. A comparison of modern sediments deposited in stratified and non-stratified waters with sediments formed since the Cambrian reveals that the ancient sea has been stratified a number of times.

THE world ocean is fully oxygenated except for some local areas in regions of upwelling or stagnation. Oxygenation is related to modern climates, that is thermo- and haloclines, which can inhibit the mixing of water masses between stratified layers but which are only temporarily stable. Furthermore, cold polar surface water sinks and moves towards the equatorial abyssal plains. These and other climate-controlled physical factors turn over the ocean's water in a matter of a few hundred to a few thousand years. At this rate, molecular oxygen is recharged much faster in the deep sea than it is consumed by the oxidation of organic matter at greater water depths.

During prolonged warmer climatic stages or when land-locked seas develop, thermo- or haloclines may become so stabilised that they only move up and down in response to seasonal changes, tectonic activities, or some other major perturbations in the environment; but they rarely break up entirely. In such conditions molecular oxygen will remain abundant in the euphotic zone but will gradually drop to zero below the density boundary. This will cause the development of a euxinic environment in which no higher forms of life can exist and molecular oxygen is replaced by hydrogen sulphide.

It is important to know what happens to the strata when this situation arises because such information is crucial to the task of explaining the origin of euxinic sediments, which are so plentiful in the stratigraphic record. Unfortunately, comparative studies between shallow and deep-water habitats are hindered because most of the marine sediments exposed on continents are of shallow-water origin and suites of abyssal sediments have only recently become available through the Deep Sea Drilling Project.

A significant aspect of this problem which has received little attention in the past concerns a possible feedback mechanism between oxidising and reducing environments which may result in the formation of specific sediment types.

We will focus attention here on the feedback question by examining sediments of the same ages and geological settings which differ only in that one group has formed below and the other above a well defined thermo-halocline. The sediments are from cores from the Black Sea and some deep East African rift lakes and they represent continuous sections through parts of the Holocene and Pleistocene. A detailed examination of a thermo-halocline, oscillating through time, will allow us to follow in slow motion the impact of the reducing on the oxidising environment and vice versa.

Phase boundaries

An heuristic theorem implies that physicochemical phenomena established at the boundary between two different states characterise these two states. In the present context the theorem implies that the physicochemical properties of the reducing and oxidising environments are 'written' on the boundary layer, that is, the thermo-halocline. We will now examine this interface.

Mechanism and rate of molecular exchange across a well