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Lee waves in a stratified flow. Part 2. Semi-circular obstacle

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Appendix

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A two-dimensional, semi-infinite, stratified shear flow in which the upstream dynamic pressure and density gradient are constant (Long's model) is considered. A complete set of *lee-wave functions*, each of which satisfies the condition of no upstream reflexion, is determined in polar co-ordinates. These functions are used to determine the lee-wave field produced by, and the consequent drag on, a semi-circular obstacle as functions of the Froude number within the range of stable flow. The Green's function (point-source solution) for the half-space also is determined in polar co-ordinates.

1. Introduction

We continue our investigation of the generation of lee waves by, and the consequent drag on, an obstacle in a two-dimensional, steady, inviscid, stratified shear flow in which the upstream dynamic pressure and density gradient are regarded as constant (Long's model). In part 1 (Miles 1968, hereinafter referred to as I, followed by the appropriate equation or section number therefrom), we considered a thin barrier in either a channel of finite height or a half-space. We consider here a semi-circular obstacle in a half-space.

Referring to figure 1, we choose a, the radius of the semi-circle as the unit of length and ar and θ as polar co-ordinates. Let U, ρ and N be the wind speed, density and intrinsic (Väisälä) frequency in the basic flow. The hypotheses that the dynamic pressure,

$$q = \tfrac{1}{2}\rho U^2, \tag{1.1}$$

and the Froude number,

$$F = U/Na \equiv 1/k, \tag{1.2}$$

are independent of elevation imply that the vertical displacement of a streamline, say $a\delta(r,\theta)$ relative to its level in the basic flow, satisfies the Helmholtz equation

$$\nabla^2\delta + k^2\delta = 0. \tag{1.3}$$

The hypotheses that the flow must pass smoothly over the boundary and that

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there must be no upstream reflexion of waves from the obstacle imply the boundary conditions

$$\delta(r,0) = \delta(r,\pi) = 0 \quad (r \geqslant 1), \tag{1.4}$$

$$\delta(1,\theta) = \sin\theta, \tag{1.5}$$

and

$$\delta(r,\theta) = o(r^{-\frac{1}{2}}) \quad (r \to \infty, \tfrac{1}{2}\pi < \theta < \pi). \tag{1.6}$$

We note that k, as defined by (1.2), is the counterpart of $\kappa = Nh/U$ in I.

We may pose the asymptotic representation of the solution to (1.3)–(1.6) in the form

$$\delta(r,\theta) \sim \mathscr{R}\{(2/\pi kr)^{\frac{1}{2}} \exp\left[i(kr - \tfrac{1}{4}\pi)\right] f(\theta) H(\tfrac{1}{2}\pi - \theta)\} \quad (kr \to \infty), \tag{1.7}$$

where $f(\theta)$ is the *complex scattering amplitude* of the far field, and H is Heaviside's step function. We then define the *differential scattering cross-section* $\sigma(\theta)$ and the *total scattering cross-section* Q as follows:†

$$\sigma(\theta) = \lim_{r\to\infty} ar\{(\nabla\delta)^2 + k^2\delta^2\} \tag{1.8a}$$

$$= (2ak/\pi)|f(\theta)|^2 \quad (0 < \theta < \tfrac{1}{2}\pi) \tag{1.8b}$$

and

$$Q = \int_0^{\frac{1}{2}\pi} \sigma(\theta)\, d\theta. \tag{1.9}$$

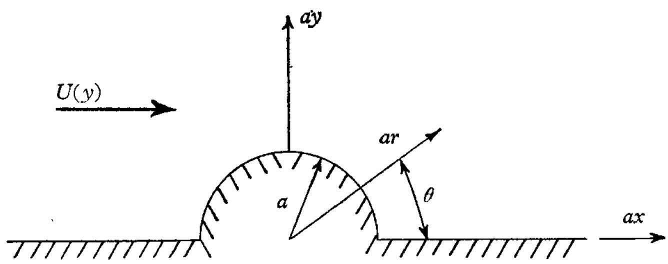

FIGURE 1. Geometrical configuration.

Our definitions are similar to, but differently normalized than, those of classica scattering theory (Morse & Feshbach 1953). The drag on the obstacle, say D, is given by the total flux of horizontal momentum, which yields [cf. I (2.7)]

$$D = qa \lim_{r\to\infty} \int_0^{\frac{1}{2}\pi} r(\delta_r^2 + k^2\delta^2)\cos\theta\, d\theta \tag{1.10a}$$

$$= q\int_0^{\frac{1}{2}\pi} \sigma(\theta)\cos\theta\, d\theta. \tag{1.10b}$$

We seek $\sigma(\theta)$, Q and D as functions of k in $0 < k < k_c$, where k_c is the minimum value of k for which density inversions ($\delta_y > 1$) appear at one or more points in the flow. It seems likely, although perhaps less than certain (since static stability may not be a necessary condition for dynamic stability of finite-amplitude motions), that Long's basic model is not physically significant for $k > k_c$. It is, of course, likely that separation would occur for $k < k_c$ in a real fluid, but this is a

† The ratio of the energy (kinetic plus potential) density in the scattered lee-wave to that in the basic flow is $\sigma(\theta)/ar$. The total wave energy diverges like $\log r$ in consequence of the assumption of constant q. This is an intrinsic deficiency of Long's model for a half-space.

separate question from that of static stability (see discussion in I). It also is possible that the instability may be local, and that a well-established lee-wave pattern may survive, for some finite range of $k > k_c$; however, Long's model could not be expected to provide reliable predictions of either the lee-wave amplitudes or the wave drag in this range.

We represent the solution to (1.3)–(1.6) as an expansion in a complete set of functions, say $\delta_n(r, \theta)$, each of which satisfies the homogeneous boundary conditions (1.4) and (1.6). We determine such a set in § 2 on the basis of the additional constraint that the singular part of δ_n behave like $a_n Y_n(kr) \sin n\theta$ as $kr \to 0$ (a_n is an appropriate constant). The resulting functions behave like $2a_n Y_n(kr) \sin n\theta$ in $\theta = (0, \frac{1}{2}\pi)$ as $kr \to \infty$, which is especially convenient for the construction of the asymptotic lee-wave field; however, they are not orthogonal in $\theta = (0, \pi)$ for fixed r, in consequence of which the expansion coefficients determined by the boundary condition (1.5) are coupled through an infinite set of linear equations. (The flat plate, as treated in I § 6, permits a solution in terms of a complete set of orthogonal functions that do exhibit a convenient asymptotic behaviour, but this appears to be a very special case.)

We determine these equations in § 3 and obtain approximate solutions by truncation. The number of equations, say N, that must be solved to obtain a given accuracy appears to increase monotonically with k. We find that the results for $N = 2$ are quite adequate for $k < 2$ and that this includes the range of principal interest, namely $0 \leqslant k \leqslant k_c \doteqdot 1{\cdot}27$. We also find that the first approximation is likely to be adequate for $k < 1$.

A difficulty that often arises with truncated expansions is the existence of singularities at which the determinant of the truncated set of equations vanishes. Such singularities may be physically significant, in the sense that they yield approximations to actual resonances or instabilities (in which case the Nth approximations to a set of critical parameters should tend to limiting values as $N \to \infty$); on the other hand, they may be, and often are, spurious consequences of the truncation.† We find that such spurious singularities exist in the present instance, but not in $k < k_c$. Stewartson (1958) appears to have experienced a similar difficulty in calculating the drag on a sphere moving along the axis of a rotating fluid. He concluded that the drag is infinite at a critical value of the Rossby number (the counterpart of our Froude number) and conjectured that the implied breakdown of the flow might be associated with, although not necessarily coincident with, the breakdown of the hypothesis of no upstream reflexion. We suggest that Stewartson's critical Rossby number may have been a spurious consequence of truncation.

An alternative, and more general, approach to the boundary-value problem posed by (1.3)–(1.6) would be to construct the Green's function (point-source

† The basic difficulty is that the convergence of the Galerkin or Ritz approximation for a non-self-adjoint system is not uniform with respect to the parameter k. Similar difficulties arise in the flutter analysis of a membrane, but not in that of a plate (Miles 1956) and led Fung (1958) to recall Courant's (1943) remark concerning the Ritz approximation: 'The first success attained by Ritz depends on his good fortune in attacking the seemingly more difficult problem...of the [vibrating] plate rather than that of the membrane.'

solution) for the half-space and then invoke Green's theorem to obtain both an integral equation for $\partial\delta/\partial n$, the normal derivative of δ on the obstacle, and a representation of δ in the exterior domain of the obstacle in terms of δ and $\partial\delta/\partial n$ on the obstacle. This approach would be circuitous for the semi-circular obstacle, but we develop the Green's fuction in §4 because of its intrinsic interest.

2. Cylindrical lee-wave functions

We seek a set of *cylindrical lee-wave functions*, say $\delta_n(r,\theta)$ $(n = 1, 2, \ldots)$, that is complete in $\theta = (0, \pi)$ for fixed r and each member of which satisfies (1.3), (1.4) and (1.6). The cylindrical wave functions for the corresponding diffraction problem, on the assumption of the time-dependence $\exp(i\omega t)$ and a radiation condition in place of (1.6), are multiples of $H_n^{(2)}(kr)\sin n\theta$ and behave like the harmonic functions $r^{-n}\sin n\theta$ as $kr \to 0$. This suggests that we pose the lee-wave functions in the form

$$\delta_n(r,\theta) = a_n\{Y_n(kr)\sin n\theta + \psi_n(r,\theta)\}, \tag{2.1}$$

where

$$a_n = -\pi(\tfrac{1}{2}k)^n/(n-1)! \tag{2.2}$$

and

$$\psi_n(r,\theta) = \sum_{q=1}^{\infty} b_{nq} J_q(kr)\sin q\theta; \tag{2.3}$$

then (1.3) and (1.4) are satisfied by each of $Y_n(kr)\sin n\theta$ and $J_q(kr)\sin q\theta$ and hence by $\delta_n(r,\theta)$, and

$$\delta_n \to r^{-n}\sin n\theta \quad (kr \to 0) \tag{2.4}$$

by virtue of our choice of a_n.

Invoking the known, asymptotic approximations to the Bessel functions in (2.1) and (2.3) and requiring the resulting approximation to δ_n to satisfy (1.6) we find that the b_{nq} must be determined such that

$$\sin 2m\theta = \sum_{p=0}^{\infty} (-)^{m+p-1} b_{2m,2p+1}\sin(2p+1)\theta \quad (n = 2m) \tag{2.5a}$$

and

$$\sin(2m+1)\theta = \sum_{p=1}^{\infty} (-)^{m+p} b_{2m+1,2p}\sin 2p\theta \quad (n = 2m+1) \tag{2.5b}$$

in $\theta = (\frac{1}{2}\pi, \pi)$. Invoking the fact that each of the sets $\sin 2m\theta$ and $\sin(2m+1)\theta$ is complete and orthogonal in both $\theta = (0, \frac{1}{2}\pi)$ and $\theta = (\frac{1}{2}\pi, \pi)$, we satisfy (2.5$a$, b) by choosing

$$\begin{aligned} b_{nq} &= (4/\pi)\,n(q^2-n^2)^{-1} \quad (n \text{ even}, q \text{ odd}) && (2.6a)\\ &= (4/\pi)\,q(q^2-n^2)^{-1} \quad (n \text{ odd}, q \text{ even}) && (2.6b)\\ &= 0 \qquad (n-q \text{ even}). && (2.6c)\end{aligned}$$

We note the alternative representation

$$\psi_n(r,\theta) = \frac{i^n}{\pi}\int_0^{\pi} \operatorname{sgn}(\tfrac{1}{2}\pi - t)\exp(-ikr\cos\theta\cos t)\sin(kr\sin\theta\sin t)\sin nt\,dt, \tag{2.7}$$

which may be inferred either from the required asymptotic behaviour of ψ_n or by expanding the integrand of (2.7) in a Fourier series in $\sin q\theta$ and integrating term by term with respect to t.

It follows from (2.3) and (2.6) that ψ_n is an even or odd function of $\frac{1}{2}\pi-\theta$ as n is even or odd, respectively; accordingly, the asymptotic cancellation of $Y_n(kr)\sin n\theta$ and $\psi_n(r,\theta)$ as $kr\to\infty$ in $\theta=(\frac{1}{2}\pi,\pi)$ implies their asymptotic identity as $kr\to\infty$ in $\theta=(0,\frac{1}{2}\pi)$, in virtue of which

$$\delta_n(r,\theta)\sim 2a_n(2/\pi kr)^{\frac{1}{2}}\sin(kr-\tfrac{1}{2}n\pi-\tfrac{1}{4}\pi)\sin n\theta \quad (kr\to\infty, 0\leqslant\theta<\tfrac{1}{2}\pi). \tag{2.8}$$

A formal proof that the δ_n defined by (2.1)–(2.3) and (2.6) form a complete set in $\theta=(0,\pi)$ can be constructed from known theorems. We rest content with the remark that an infinite expansion in $\delta_n(r,\theta)$ may be expressed as a Fourier series in $\sin q\theta$, subject only to rather mild restrictions on convergence.

The set δ_n is not orthogonal. An orthogonal set of the form

$$\hat{\delta}_n(r,\theta)=\delta_n(r,\theta)+\sum_{s=1}^{n-1}C_{ns}\delta_s(r,\theta) \tag{2.9}$$

could be constructed by the Schmidt orthogonalization procedure, but the r-dependence of $\hat{\delta}_n(r,\theta)$ would appear to be far too complicated to be offset by any advantage of orthogonality for the problems at hand.

3. Solution for semi-circular obstacle

We pose the solution of (1.3)–(1.6) in the form

$$\delta(r,\theta)=\sum_1^\infty d_n(k)\,\delta_n(r,\theta). \tag{3.1}$$

Since δ_n satisfies (1.3), (1.4) and (1.6), it remains only to determine the d_n to satisfy (1.5).

Substituting δ_n, as given by (2.1) and (2.3), into (3.1) and reversing the order of summation in the resulting double sum, we place the result in the form

$$\delta(r,\theta)=\sum_{n=1}^\infty\left\{\sum_{q=1}^\infty d_q Z_{nq}(r)\right\}\sin n\theta, \tag{3.2}$$

where
$$Z_{nq}(r)=a_q\{\delta_{nq}Y_n(kr)+b_{qn}J_n(kr)\}, \tag{3.3}$$

and δ_{nq} is the Kronecker delta. Substituting (3.2) into (1.5) and equating coefficients of $\sin n\theta$, we obtain the infinite set of linear equations

$$\sum_{q=1}^\infty Z_{nq}(1)\,d_q=\delta_{1n}\quad (n=1,2,\ldots). \tag{3.4}$$

We observe that

$$Z_{nq}(1)=\delta_{nq}a_nY_n(k)+O(k^{n+q}) \tag{3.5a}$$

$$=\delta_{nq}\{1+O(k^2\log k)\}+O(k^{n+q})\quad (k\to 0). \tag{3.5b}$$

Turning to the calculation of the scattering amplitude, $f(\theta)$, we substitute (2.8) into (3.1) and equate the result to (1.7) to obtain

$$f(\theta)=\pi k\sum_{n=1}^\infty(-i)^{n-1}F_n(k)\sin n\theta, \tag{3.6}$$

where
$$F_n(k)\equiv a_nd_n/a_1=(\tfrac{1}{2}k)^{n-1}d_n(k)/(n-1)!. \tag{3.7}$$

Substituting (3.6) into (1.8b), we obtain

$$\frac{\sigma(\theta)}{2\pi a k^3} \equiv \sigma_*(\theta)\sin^2\theta$$

$$= \left[\left\{\sum_{m=0}^{\infty}(-)^m F_{2m+1}\sin(2m+1)\theta\right\}^2 + \left\{\sum_{m=1}^{\infty}(-)^m F_{2m}\sin 2m\theta\right\}^2\right]. \quad (3.8)$$

Substituting (3.8) into (1.9) and (1.10b), we place the results in the normalized forms

$$\frac{Q}{\frac{1}{2}\pi^2 a k^3} \equiv Q_*(k) = \sum_1^{\infty} F_n^2 \quad (3.9)$$

and

$$\frac{D}{\frac{2}{3}\pi q a k^3} \equiv D_*(k) = \tfrac{3}{2}\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}\left\{\frac{1}{1-4(m-p)^2} - \frac{1}{4(m+p+1)^2-1}\right\}F_{2m+1}F_{2p+1}$$

$$+\tfrac{3}{2}\sum_{m=1}^{\infty}\sum_{p=1}^{\infty}\left\{\frac{1}{1-4(m-p)^2} + \frac{1}{4(m+p)^2+1}\right\}F_{2m}F_{2p}. \quad (3.10)$$

We have renormalized the expansion coefficients (from d_n to F_n) such that each of $\sigma_*(\theta)$, Q_* and D_* is $1+O(k^2\log k)$ as $k\to 0$.

Truncating (3.4) at $n = N = 1$, we obtain the first approximations

$$\sigma_*(\theta) = Q_*(k) = D_*(k) = \{F_1^{(1)}(k)\}^2 \quad (N=1), \quad (3.11)$$

where
$$F_1^{(1)}(k) = -\{\tfrac{1}{2}\pi k Y_1(k)\}^{-1}. \quad (3.12)$$

Truncating (3.2) at $n = N = 2$, we obtain the second approximations

$$\{\sigma_*(\theta), Q_*(k), D_*(k)\} = \{F_1^{(2)}(k)\}^2[1+H(k)\{4\cos^2\theta, 1, \tfrac{8}{5}\}] \quad (N=2), \quad (3.13)$$

where
$$F_1^{(2)}(k) = F_1^{(1)}(k)\{1+(64J_1J_2/9\pi^2Y_1Y_2)\}^{-1} \quad (3.14a)$$

$$= F_1^{(1)}(k)\{1-\tfrac{1}{18}k^6+O(k^8\log k)\}, \quad (3.14b)$$

$$H(k) = (8J_2/3\pi Y_2)^2 \quad (3.15a)$$

$$= O(k^8), \quad (3.15b)$$

and the argument of each of the Bessel functions is k. We infer from (3.13)–(3.15) that σ_*, Q_* and D_* differ from one another by $O(k^8)$ and that the error factor for the first approximation of (3.11) is $1+O(k^6)$ as $k\to 0$. The first approximation breaks down at the smallest zero of $Y_1(k)$, $k \doteqdot 2{\cdot}2$, where $F_1^{(1)} = \infty$; the nth approximation breaks down at $k = k_n$, where k_n is fairly close to the smallest zero of $Y_n(k)$.† Invoking a known result for the zeros of $Y_n(k)$, we infer that k_n is a monotonically increasing function of n. The nth-order, truncated determinant $|Z_{ij}|$ vanishes at $k = k_n$, thereby yielding infinite values of $F_1, F_2, \ldots, F_n$, except for $n = 2$. The second approximation is anomalous in that it yields finite values of F_1 and F_2 near the first zero of Y_2, $k \doteqdot 3{\cdot}4$; nevertheless, it must be regarded as inaccurate for k near or greater than this zero.

A rough estimate of k_c, the value of k at which the constraint of static stability, $\delta_y < 1$, is first violated, may be determined from the first approximation, namely

$$\delta = F_1^{(1)}(k)\,\delta_1(r,\theta) \quad (N=1), \quad (3.16)$$

† I am indebted to my student Herbert Huppert for this suggestion.

which yields $k_c \doteqdot 1{\cdot}2$. Huppert (appendix) uses the second approximation to obtain $k_c = 1{\cdot}27$ and presents streamline patterns for $k < k_c$, $k = k_c$ and $k > k_c$.

The second approximations to Q_* and D_*, as given by (3.13), are plotted in figure 2; D_* differs from Q_* by less than 1 % in $k < 1{\cdot}2$. The first approximations of (3.11) differ from the second approximations by less than 5 % in $k < 1$, but depart rapidly therefrom in $k > 1$ (14 % at $k = 1{\cdot}2$). The third approximations to Q_* and D_* (not given explicitly above) differ from the second approximations by 1 % at $k = 2$. We conclude that the second approximations should be adequate, and that the difference between D_* and Q_* is small, in the range of stable flow, $0 \leqslant k \leqslant k_c \doteqdot 1{\cdot}27$.

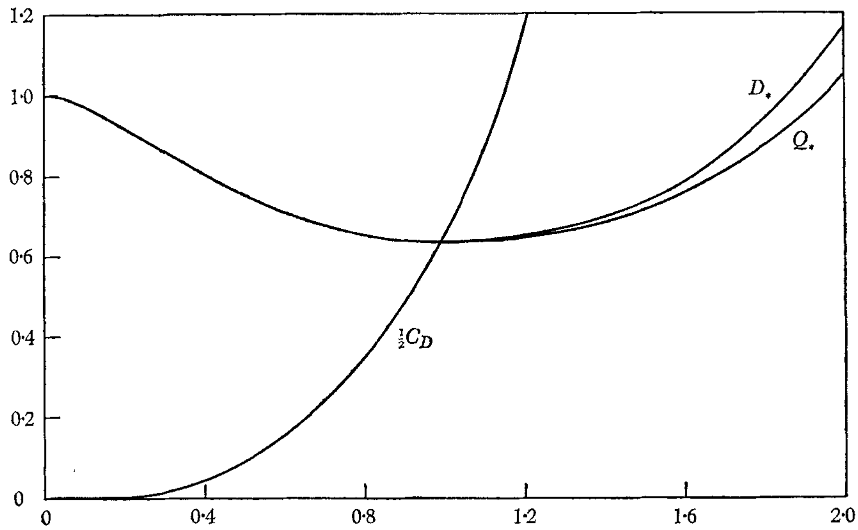

FIGURE 2. Normalized drag (D_*), total scattering cross-section (Q_*), and drag coefficient for semi-circular obstacle, as given by (3.13) and (3.17). The flow may be unstable for $k > k_c \doteqdot 1{\cdot}27$.

The drag coefficient
$$\tfrac{1}{2}C_D = D/\rho U^2 a \tag{3.17}$$
also is plotted in figure 2. We conclude that the drag coefficient C_D for *unseparated* flow over a semi-circular obstacle is not likely to exceed 2·6 in the range of stable flow. We again emphasize that the question of stability, as treated here, is distinct from the question of viscous separation, which may occur for $k < k_c$ in a real fluid.

4. Green's function for half-space

The Green's function, or point-source solution, for a half-space, is determined by
$$\nabla^2 G + k^2 G = -\hat{\delta}(R), \tag{4.1}$$
$$G = 0 \quad (y = 0), \tag{4.2}$$
and
$$G = o(r^{-\frac{1}{2}}) \quad (r \to \infty, \tfrac{1}{2}\pi < \theta < \pi), \tag{4.3}$$
where $\hat{\delta}$ is Dirac's delta function (we reserve δ for streamline displacement), and
$$R = |\mathbf{r} - \boldsymbol{\rho}| = \{(x-\xi)^2 + (y-\eta)^2\}^{\frac{1}{2}} = \{r^2 + \rho^2 - 2r\rho\cos(\theta - \phi)\}^{\frac{1}{2}} \tag{4.4}$$

is the distance from $\boldsymbol{\rho} = (\xi, \eta)$ to $\mathbf{r} = (x, y)$ or, in polar co-ordinates, from (ρ, ϕ) to (r, θ).†

A particular solution of (4.1) is given by

$$-\tfrac{1}{4}Y_0(kR) = -\tfrac{1}{4}\sum_{n=0}^{\infty}(2-\delta_n^0)J_n(k\rho)Y_n(kr)\cos n(\theta-\phi) \quad (\rho \leqslant r), \tag{4.5}$$

wherein ρ and r must be interchanged if $\rho > r$. Changing the sign of ϕ in (4.5), we obtain a complementary solution of (4.1), say $-\frac{1}{4}Y_0(kR_*)$, corresponding to the image of (4.5) in the plane $y = 0$. Subtracting this solution from that of (4.5), we obtain

$$G_1(\mathbf{r}, \boldsymbol{\rho}) = -\tfrac{1}{4}Y_0(kR) + \tfrac{1}{4}Y_0(kR_*) \tag{4.6a}$$

$$= -\sum_1^{\infty} J_n(k\rho)Y_n(kr)\sin n\theta \sin n\phi \quad (\rho < r), \tag{4.6b}$$

which satisfies (4.1) and (4.2). Referring to (2.1) and recalling that $\psi_n(r, \theta)$ satisfies (1.3) and (1.4), we find that

$$G_2(\mathbf{r}, \boldsymbol{\rho}) = -\sum_1^{\infty} J_n(k\rho)\psi_n(r, \theta)\sin n\phi \tag{4.7}$$

is a complementary solution of (4.1) and (4.2) that, when added to G_1, yields a solution of (4.1)–(4.3), namely

$$G(\mathbf{r}, \boldsymbol{\rho}) = -\sum_1^{\infty} J_n(k\rho)\{Y_n(kr)\sin n\theta + \psi_n(r, \theta)\}\sin n\phi \tag{4.8a}$$

$$= -\sum_1^{\infty} a_n^{-1} J_n(k\rho)\delta_n(r, \theta)\sin n\phi. \tag{4.8b}$$

We obtain an alternative representation of G_2 by substituting the series representation of ψ_n, as given by (2.3) and (2.6), into (4.8) and reversing the order of summation:

$$G_2(\mathbf{r}, \boldsymbol{\rho}) = \sum_1^{\infty} J_n(kr)\psi_n(\rho, \phi)\sin n\theta \tag{4.9a}$$

$$\equiv -G_2(\boldsymbol{\rho}, \mathbf{r}). \tag{4.9b}$$

Adding (4.9a) to (4.6b), we obtain

$$G(\mathbf{r}, \boldsymbol{\rho}) = \sum_{n=1}^{\infty}\{\psi_n(\rho, \phi)J_n(kr) - J_n(k\rho)Y_n(kr)\sin n\phi\}\sin n\theta \quad (\rho \leqslant r). \tag{4.10}$$

Invoking the asymptotic approximation (2.8) in (4.8b), we obtain

$$G(\mathbf{r}, \boldsymbol{\rho}) \sim \mathscr{R}\{(2/\pi kr)^{\frac{1}{2}}\exp[i(kr - \tfrac{1}{4}\pi)]g(\theta, \rho, \phi)H(\tfrac{1}{2}\pi - \theta)\} \quad (r \to \infty), \tag{4.11}$$

where

$$g = 2\sum_1^{\infty}(-i)^{n-1}J_n(k\rho)\sin n\theta \sin n\phi \tag{4.12a}$$

$$= \exp(-ik\xi\cos\theta)\sin(k\eta\sin\theta). \tag{4.12b}$$

† The density, denoted by ρ in § 1, does not arise explicitly in the subsequent analysis, where ρ appears only as a radius.

We may base a formulation of the boundary-value problem for an arbitrarily shaped obstacle on $G(\mathbf{r},\boldsymbol{\rho})$. Let $\delta(\mathbf{r})$ satisfy (1.3), (1.4) outside of an obstacle of prescribed contour C, and (1.6). Applying Green's second theorem to the functions $\delta(\boldsymbol{\rho})$ and $G(\mathbf{r},\boldsymbol{\rho})$ around a closed contour made up of C, the plane $y = 0$ outside of C, and the semicircle $\rho = \infty$, we obtain

$$\delta(\mathbf{r}) = \int_C \left\{\delta(\boldsymbol{\rho})\frac{\partial}{\partial n}G(\mathbf{r},\boldsymbol{\rho}) - G(\mathbf{r},\boldsymbol{\rho})\frac{\partial}{\partial n}\delta(\boldsymbol{\rho})\right\} dl(\boldsymbol{\rho}), \tag{4.13}$$

where $\mathbf{n} \equiv \mathbf{n}(\boldsymbol{\rho})$ is the outwardly directed normal to C, $dl(\boldsymbol{\rho})$ is a differential element of C, and the integration is in the counterclockwise sense. Invoking the boundary condition $\delta = y$ on C then yields an integral equation for $\partial\delta/\partial n$ on C, the solution of which would permit the calculation of $\delta(\mathbf{r})$ from (4.13).

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Appendix

By HERBERT E. HUPPERT

We present some additional, numerical results. Retaining the first two lee-wave functions in (3.1) and evaluating δ_y numerically, we find that instability first occurs at $r_c = 4{\cdot}0$, $\theta_c = 57°$ for $k = k_c = 1{\cdot}27$.

The contour lines of the function

$$\psi = y - \delta(r,\theta)$$

are the streamlines of the flow. Four flow patterns are shown in figures A1–A4 (the contour interval is 0·7).

Long (1955) determined flow patterns for the stratified flow over an obstacle in a channel, for which the lee-wave spectrum is discrete, rather than continuous. The flow patterns in the half-space and the channel are similar if one and only one mode is present in the channel; they are quite different if either no, or more than one, mode is present, demonstrating the marked influence of the upper boundary condition.

Experiments performed by Long indicate that, although there is local instability in regions where $\delta_y > 1$, the downstream wave may be similar to that presented in figure A4 for $k = 1{\cdot}5 > k_c$.

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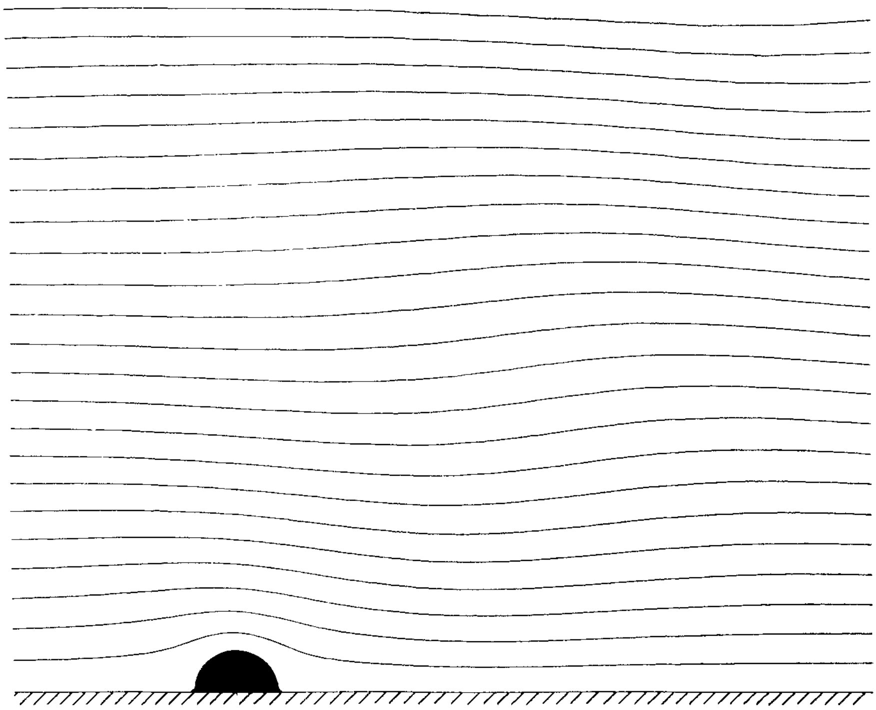

FIGURE A 1. Stratified shear flow over a semi-circular obstacle for $k = 0{\cdot}5$.

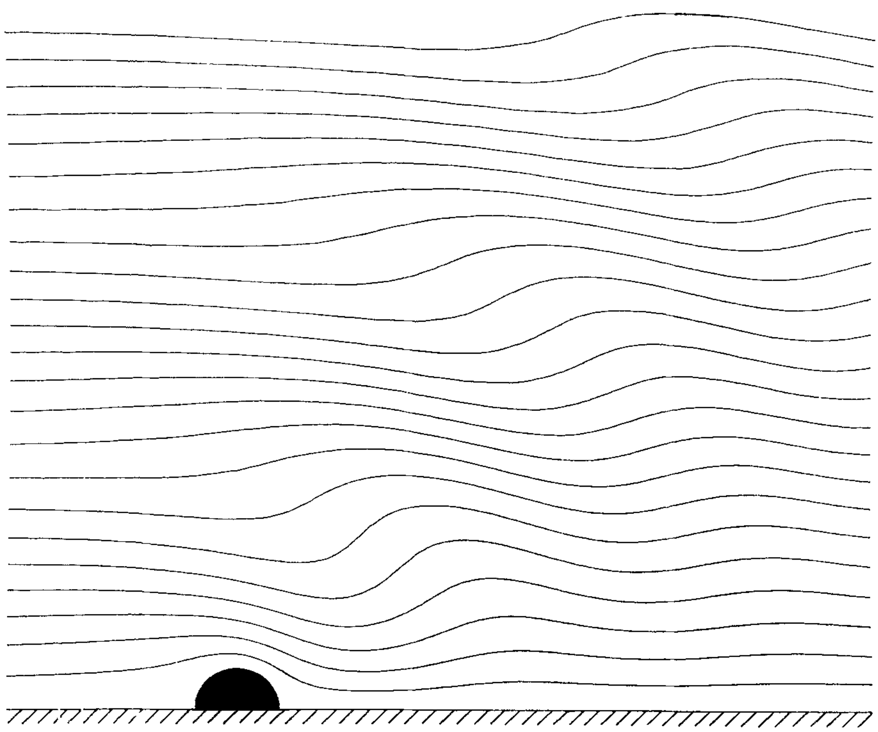

FIGURE A 2. Stratified shear flow over a semi-circular obstacle for $k = 1{\cdot}0$.

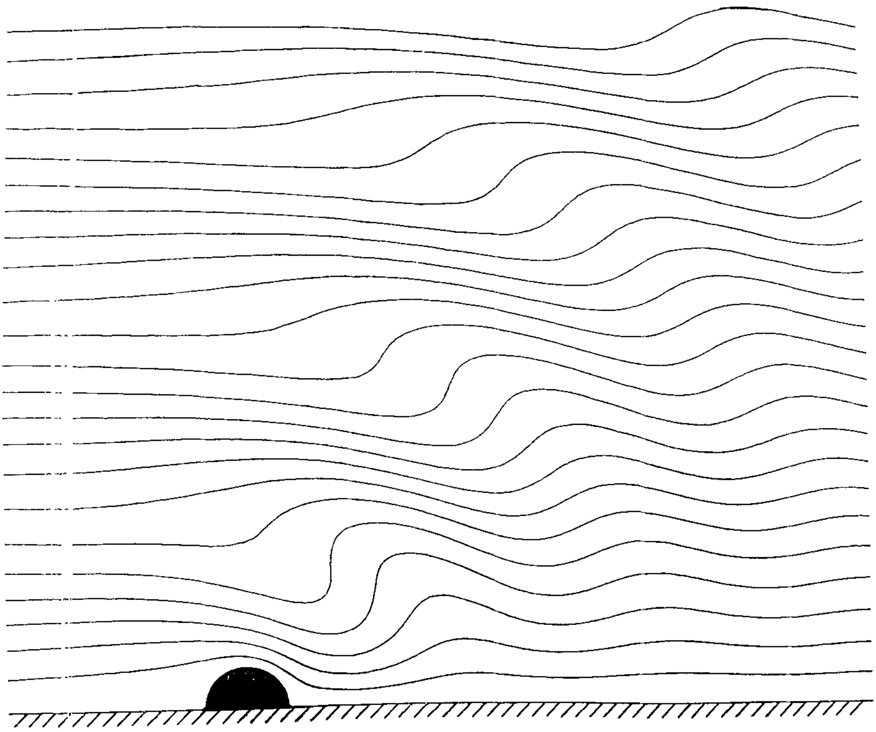

FIGURE A3. Stratified shear flow over a semi-circular obstacle for $k = k_c = 1{\cdot}27$.

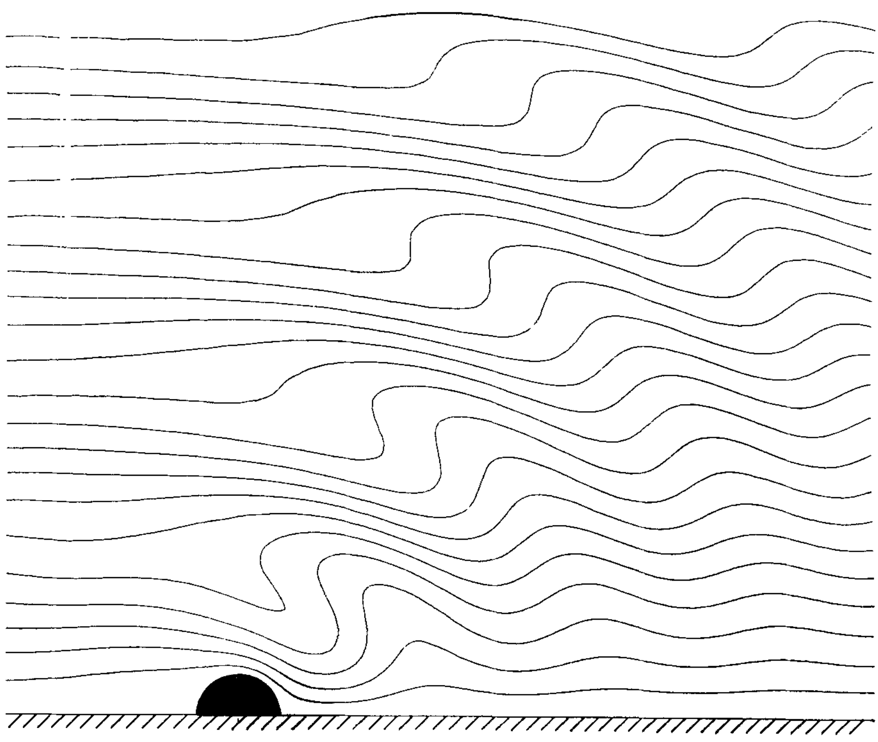

FIGURE A4. Stratified shear flow over a semi-circular obstacle for $k = 1{\cdot}5$.

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